

# THE ASYMPTOTIC THEORY OF THE POVERTY INTENSITY IN VIEW OF EXTREME VALUE THEORY FOR TWO SIMPLE CASES.

GANE SAMB LO AND SERIGNE TOUBA SALL

ABSTRACT. Let  $Y_1, Y_2, \dots$  be independent observations of the income variable of some population, with underlying distribution  $G$ . Given a poverty line  $Z$ , for each  $n \geq 1$ ,  $q_n$  is the number of poors in the population. The general form of poverty measures used by economists to monitor the welfare evolution of this population is

$$P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^q c(n, q, j) d\left(\frac{Z - Y_{j,n}}{Z}\right).$$

This class includes the most popular poverty measures like the Sen, Shorrocks and Greer-Foster-Thorbecke statistics. We give a complete asymptotic normality theory in the frame of extreme value theory. In this paper, the two simple cases of the Pareto and exponential distributions are studied with the intensity poverty. Simulations and applications to the senegalese data are available.

## Résumé

Soit  $Y_1, Y_2, \dots$  des observations indépendantes de la variable revenu d'une population donnée. Etant donné un seuil de pauvreté  $Z$ , alors  $q_n$  est pour tout  $n \geq 1$  le nombre de pauvres dans la population. La forme générale des indicateurs de pauvreté est

$$P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^q c(n, q, j) d\left(\frac{Z - Y_{j,n}}{Z}\right).$$

Cette classe de mesures contient les indicateurs classiques tels que ceux de Sen, Shorrocks et de Greer-Foster-Thorbecke. Dans cet article, nous préparons la théorie asymptotique complète de ces indicateurs dans le cadre des valeurs extrêmes. Nous nous concentrons ici sur l'intensité de la pauvreté pour deux cas simples : La loi de Paréto et la loi exponentielle. Des simulations et des applications sur les données du Sénégal sont disponibles.

## 1. INTRODUCTION.

Poverty is measured in Economics by statistics of the general form

$$(1.1) \quad P_N = \frac{1}{a(Q)b(N)} \sum_{j=1}^Q c(N, Q, j) d\left(\frac{Z - Y_{j,N}}{Z}\right)$$

where  $a, b, c, d$  are given functions,  $Q$  the number of poors in the studied population  $P$  of size  $N$ ,  $Z$  the poverty line and  $Y_{1,N} \leq Y_{2,N} \leq \dots \leq Y_{N,N}$  are the ordered incomes of the individuals of  $P$ . The poverty line  $Z$  is defined by

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economics specialists or governmental authorities so that any individual or household with income (say yearly) less than  $Z$  is considered as poor. The economic welfare of  $P$  is monitored both by the measures (1.1) and other measures of the inequality of the income distribution.

A very large class of poverty measures can be found in the literature. One may divide them into two classes. The first includes the non weighted ones, for which  $c(N, Q, j) \equiv 1$ . The most popular of them is the Foster-Greer-Thorbecke class which is, for  $\alpha \geq 0$ ,

$$(1.2) \quad FGT(\alpha) = \frac{1}{N} \sum_{j=1}^Q \left( \frac{Z - Y_{j,N}}{Z} \right)^\alpha.$$

For  $\alpha = 0$ , (1.2) reduces to  $Q/N$ , the headcount of poors while for  $\alpha = 1$  and  $\alpha = 2$ , it is respectively interpreted as the severity of poverty and the depth in poverty. The second class consists of the weighted measures. We mention here two of its famous members, the Sen measure

$$(1.3) \quad P_{SE,N} = \frac{2}{N(Q+1)} \sum_{j=1}^Q (Q-j+1) \left( \frac{Z - Y_{j,N}}{Z} \right)$$

and the Shorroks(1995) one

$$(1.4) \quad P_{SH,N} = \frac{1}{N^2} \sum_{j=1}^Q (2N - 2j + 1) \left( \frac{Z - Y_{j,N}}{Z} \right).$$

These discrete statistics have been widely prospected in the frame of poverty reduction. The interested readers are referred to [13], [9], [10], [15], [19].

We now want to settle an asymptotic theory of the statistics (1.1). We suppose that we have independent and identically observations of the income  $Y_1, Y_2, \dots$  with underlying distribution  $G$  with lower endpoint  $y_0 = \inf\{x, G(x) > 0\} \geq 0$ . Given a poverty line  $Z$ , we have the (random) number of poors  $q_n = q$  in the sample of size  $n$  :  $Y_1, Y_2, \dots, Y_n$ . The sample measure of poverty becomes

$$(1.5) \quad P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^q c(n, q, j) d \left( \frac{Z - Y_{j,n}}{Z} \right).$$

Clearly, this depends on the  $q$  lower extreme values and suggests to use the extreme value theory to handle such statistics. If  $q/n \rightarrow 0$ , the study will only need the lower tail of  $G$ . But for many poor countries  $q/n$  is greater than 50% so that the reasonable hypothesis must be  $q/n \rightarrow \xi = G(Z) \in ]0, 1[$ . Thus, at most the lower tail and the center of  $G$  will be used. This is in conformity with the focalization hypothesis of poverty measure which says that any poverty statistics is strictly function of the poors income. The law of such statistics are usually guided by the extremal domain of attraction of the sequence of minima  $Y_{1,n}$ ,  $n \geq 1$ . It happens that much of the theory is set up by using the extremal domain of attraction of the maxima. We make a brief recall of our needs in extreme value theory using maxima in Section 2 . In Section 3, we state our results for the intensity poverty whose the

asymptotic theory is completely described for the Pareto and the exponential cases.

## 2. BASICS OF EXTREME VALUES THEORY USING MAXIMA

Let  $Y_1, Y_2, \dots$  be a sequence of independent and identically distributed with  $P(Y_1 \leq x) = G(x)$  and  $G(0) = 0$ . The estimation of the law of the maximum  $Y_{n,n} = \max(Y_1, \dots, Y_n)$  is one of the most important questions of the extreme value theory. Recall that  $Y_{n,n}$  converges in type to some random variable  $Z$  with non degenerated distribution function  $H(\cdot)$  if and only if there exists two sequences of real numbers  $(a_n > 0, n \geq 1)$  and  $(b_n, n \geq 1)$  such that

$$(2.1) \quad \lim_{n \rightarrow \infty} \mathbb{P}(Y_{n,n} \leq a_n x + b_n) = \lim_{n \rightarrow \infty} G^n(a_n x + b_n) = H(x), \quad \forall x \in \mathbb{R}.$$

It is then said that  $G$  is in the domain of attraction of  $H$ , denoted by  $G \in D(H)$ . It is well-known that  $H$  is necessarily of these three types :

$$(2.2) \quad H(x) = \varphi_\alpha(x) = \exp(-x^{-\alpha}) I_{(x \geq 0)}; \quad (\text{type I})$$

$$(2.3) \quad H(x) = \Lambda(x) = \exp(-e^{-x}); \quad (\text{type II})$$

$$(2.4) \quad H(x) = \psi_\gamma(x) = \exp(-(-x)^\gamma) I_{(x \leq 0)} + I_{(x > 0)}; \quad \gamma > 0; \quad (\text{type III})$$

where  $I_A$  is indicator function of the set  $A$ . It should be noticed that this function  $H$  represents an equivalence class of the following binary relation between the distribution functions :

$$(2.5) \quad H_1 \mathcal{R} H_2 \Leftrightarrow (\exists (a, b) \in \mathbb{R}_+^* \times \mathbb{R}, \forall x \in \mathbb{R}, H_1(x) = H_2(ax + b))$$

In this case, any of the function  $H$  in (2.2), (2.3) and (2.4) may be represented as

$$H(x) = H_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \quad \text{for } 1 + \gamma x > 0,$$

where  $(1 + \gamma x)^{-1/\gamma}$  is interpreted as  $e^{-x}$  for  $\gamma = 0$ . The values  $\gamma > 0$ ,  $\gamma = 0$  et  $\gamma < 0$  are respectively related to the types I, II et III.

The class of the whole domain of attraction (the set of all distribution functions attracted to some  $H$ ) is denoted by  $\mathfrak{D} = \cup_{\alpha > 0} \cup_{\gamma > 0} D(\varphi_\alpha) \cup D(\psi_\gamma) \cup D(\Lambda)$ .

The interested reader is referred to [5], [7] or [16] for a general introduction to extreme value theory. The characterization of these three domains received very much consideration in the two past decades. The frame of extreme value theory has been widely used to find the asymptotic normality of sums of extreme values as well as Hill type statistics (see [12]) mainly [14], [17], [6], [11].

Theses representations are then used for the generalized inverse of  $G$  :  $G^{\leftarrow}(s) = \inf\{x, G(x) \geq s\}$ ,  $0 \leq s \leq 1$ . We have first for  $G \in D(\varphi_\gamma)$ ,  $z_0 = \sup\{x, G(x) < 1\} = +\infty$  and

$$(2.6) \quad G^{\leftarrow}(1-u) = c(1+f(u)) u^{1/\gamma} \exp\left(\int_u^1 b(t)t^{-1}dt\right), \quad 0 < u < 1,$$

Next for  $G \in D(\psi_\gamma)$ ,  $z_0 = \sup\{x, G(x) < 1\} < +\infty$  and

$$(2.7) \quad z_0 - G^{\leftarrow}(1-u) = c(1+f(u)) u^{1/\gamma} \exp\left(\int_u^1 b(t)t^{-1}dt\right), \quad 0 < u < 1$$

and finally, for  $G \in D(\Lambda)$ ,  $z_0 = \sup\{x, G(x) < 1\} = +\infty$  or  $< \infty$ ,

$$(2.8) \quad G^{\leftarrow}(1-u) = d - s(u) + \int_u^1 s(t)t^{-1}dt, \quad 0 < u < 1,$$

where  $s(u) = c(1+f(u)) \exp(\int_u^1 b(t)t^{-1}dt)$ ,  $0 < u < 1$ . In each of these formulae,  $(f(u), b(u)) \rightarrow (0,0)$  when  $u \rightarrow 0$  while  $c$  and  $d$  are constants. (2.6) and (2.7) are the Karamata representations while (2.8) is that of de Haan-Mason-Deheuvels.

To finish, recall the characterization of Theorem 2.4.3 in [7]

First  $G \in D(\Lambda)$  iff

$$(2.9) \quad \frac{1 - G(t + xr(t))}{1 - G(t)} \rightarrow e^{-x}, \quad \text{as } x \rightarrow z_0$$

for any  $x$  and for some positive function  $r(\cdot)$ . Moreover, all the functions  $r(\cdot)$  in (2.9) are equivalent between them as  $x \rightarrow z_0$ . Further, by Theorem 2.4.1 in [6],  $G \in D(\Lambda)$  iff

$$(2.10) \quad \forall x > 0, \{G^{\leftarrow}(1-ux) - G^{\leftarrow}(1-u)\}/r(u) \rightarrow -\log x,$$

for some positive function  $r(\cdot)$ . Notice that  $z_0$  may be finite or not. Next,  $G \in D(\varphi_\gamma)$  iff  $z_0 = +\infty$  and

$$(2.11) \quad \forall \lambda > 0, (1 - G(\lambda x))/(1 - G(x)) \rightarrow \lambda^{-\gamma}.$$

In this case,  $G$  admits the Karamata representation

$$(2.12) \quad 1 - G(x) = c(1+f_1(x)) x^{-\gamma} \exp\left(\int_1^x b_1(t)t^{-1}dt\right), \quad y_0 < x < z_0,$$

where  $(f_1(u), b_1(u)) \rightarrow (0,0)$  as  $u \rightarrow 0$ . The function  $L(x) = c(1+f_1(u)) \exp(\int_1^x b_1(t)t^{-1}dt)$  is the representation of a function slowly varying at infinity that is  $L(\lambda x)/L(x) \rightarrow 1$  as  $x \uparrow \infty$  for any  $\lambda > 1$ . Finally  $G \in D(\psi_\gamma)$  iff ( $z_0 < \infty$  and  $G(z_0 - 1/\cdot) \in D(\varphi_\gamma)$ ).

Now, we must remark in our frame that  $Y$  is an income variable. Then its lower endpoint  $y_0$  is not negative. This allows us to study (1.5) via the transform  $X = 1/(Y - y_0)$ . Throughout this paper, the distribution function of  $X$  is  $F$ , whose upper endpoint is then  $+\infty$ . Its extremal domain is  $D(\Lambda)$  or  $D(\varphi_\gamma)$ . Hence (1.5) is transformed as

$$(2.13) \quad P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^q c(n, q, j) d\left(\frac{Z - y_0 - X_{n-j+1, n}^{-1}}{Z}\right),$$

and studied following the extremal law of  $X$  using the general representations (2.6) and (2.8).

The aim of this paper is to prepare the general asymptotic theory of the poverty measures by giving the complete study of one represent of each of the possible extremal domain : The Pareto distribution of parameter  $\gamma$  (for the type I) and the exponential distribution (for the type II). Only the non weighted measures are also concerned.

Troughout the paper, we use the representations of the studied random variables  $X_i, i \geq 1$ , by  $F^{-1}(1 - U_i), i \geq 1$ , where  $U_1, U_2, \dots$  is a sequence of independent random variables uniformly distributed on  $(0, 1)$ . It follows that :

$$(2.14) \quad \{ X_{n-i+1,n}, 1 \leq i \leq n ; n \geq 1 \} \stackrel{d}{=} \{ F^{-1}(1 - U_{i,n}), 1 \leq i \leq n ; n \geq 1 \}$$

where  $U_{1,n} < \dots < U_{n,n}$  are the order statistics based on  $U_1, \dots, U_n$ .

Our best achievement is the complete asymptotic characterization of the poverty measure

$$(2.15) \quad I_n = \frac{1}{n} \sum_{j=1}^q \left( \frac{Z - Y_{j,n}}{Z} \right)$$

for the exponential distribution of parameter  $\lambda$ ,

$$(2.16) \quad F(x) = \begin{cases} 1 - \exp(-\lambda x), & \lambda > 0, \text{ for } x > 0 \\ 0 & \text{else} \end{cases}$$

and for the Paréto distribution of parameter  $\gamma > 0$ , i. e.,

$$(2.17) \quad F(x) = \begin{cases} 1 - x^{-1/\gamma}, & \text{for } x > 1 \\ 0 & \text{else} \end{cases}$$

Here are our results.

### 3. RESULTS

**Theorem 1.** *Let  $F = 1 - G(y_0 + 1/\cdot)$  be the exponential distribution function of parameter  $\lambda > 0$ , then  $\sqrt{n}(I_n - C_n) \rightarrow N(0, \sigma^2)$  where*

$$C_n = \frac{1}{nZ} q(Z - y_0) + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{q/n} \frac{1}{\log(s)} ds$$

and

$$\left( \frac{Z}{\lambda} \right)^2 \sigma^2 = \frac{-2}{\log G(Z)} \int_0^{G(Z)} (\log s)^{-2} ds + 2 \int_0^{G(Z)} (\log s)^{-3} - \left( \int_0^{G(Z)} \frac{dt}{(\log t)^2} \right)^2$$

with  $G(Z) = \exp(-\lambda/(Z - y_0))$  for  $Z > 0$ .

**Theorem 2.** *Let  $F=1-G(y_0+1/\cdot)$  be the Pareto distribution function of parameter  $\gamma^{-1} > 0$ , then  $\sqrt{n}(I_n - D_n) \rightarrow N(0, \theta^2)$  with*

$$D_n = \frac{1}{nZ}q(Z - y_0) - \frac{1}{Z(\gamma + 1)}((q/n)^{\gamma+1} - n^{-\gamma-1})$$

and

$$\theta^2 = (\gamma/Z)^2 G(Z)^{2\gamma+1} \left( \frac{2}{(\gamma + 1)(2\gamma + 1)} - \frac{G(Z)}{(\gamma + 1)^2} \right)$$

with  $G(Z) = ((Z - y_0)^{\frac{1}{\gamma}} \wedge 1)$ .

#### 4. PROOFS

In the beginning we use a special representation of

$$(4.1) \quad I_n = \frac{1}{n} \sum_{j=1}^q \left( \frac{Z - Y_{j,n}}{Z} \right)$$

By using the transform  $X=1/(Y-y_0)$ , we get

$$(4.2) \quad I_n = \frac{1}{nZ} \sum_{j=1}^q \left( Z - y_0 - \frac{1}{X_{n-j+1,n}} \right)$$

$$(4.3) \quad I_n = \frac{1}{nZ}q(Z - y_0) - \frac{1}{nZ} \sum_{j=1}^q \frac{1}{X_{n-j+1,n}}$$

$$(4.4) \quad = \frac{1}{nZ}q(Z - y_0) - \frac{1}{nZ} \sum_{j=1}^q n \int_{\frac{n-j}{n}}^{\frac{n-j+1}{n}} \frac{1}{Q_n(s)} ds,$$

where  $Q_n$  is the empirical quantile function based on  $X_1, X_2, \dots, X_n$  defined by  $Q_n(s) = X_{j,n}$  for  $\frac{j}{n} < s \leq \frac{j+1}{n}$  for  $j=1, \dots, n$  and  $Q_n(0) = X_{1,n}$ . Furthermore, we get by (2.14) the following representation of  $Q_n$  via the empirical quantile function  $V_n$  based on  $U_1, U_2, \dots, U_n$  :

$$(4.5) \quad \{Q_n(t), 0 \leq t \leq 1\} =^d \{F^{\leftarrow}(V_n(t)), 0 \leq t \leq 1\}$$

$$(4.6) \quad =^d \{F^{\leftarrow}(1 - V_n(t))\},$$

where  $F^{\leftarrow}(s) = \inf\{x, F(x) \geq s\}$ ,  $0 \leq s \leq 1$ , is the generalized function of  $F$ . Recall that  $G_n(x) = q/n$  when  $G_n$  denotes the empirical distribution function based on  $Y_1, Y_2, \dots, Y_n$ . Thus, the final representation of  $I_n$  is

$$(4.7) \quad I_n = \frac{1}{nZ}q(Z - y_0) - \frac{1}{nZF^{\leftarrow}(1 - U_{1,n})} - \frac{1}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{1}{F^{\leftarrow}(1 - V_n(s))} ds.$$

Now we are ready to prove the two theorems.

Proof of theorem 1.

We have  $F^-(s) = -\log(1-s)/\lambda, 0 \leq s < 1$  and from (4.7),

$$(4.8) \quad I_n = \frac{1}{nZ}q(Z - x_0) + \frac{\lambda}{nZ \log(U_{1,n})} + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{1}{\log(V_n(s))} ds.$$

Put

$$(4.9) \quad C_n = \frac{1}{nZ}q(Z - x_0) + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{1}{\log(s)} ds$$

By the mean value theorem

$$(4.10) \quad I_n - C_n = \frac{\lambda}{nZ \log(U_{1,n})} - \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{V_n(s) - s}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds,$$

where  $\zeta_n(s) \in [\min(s, V_n(s)); \max(s, V_n(s))]$ . Thus

$$(4.11) \quad \sqrt{n}(I_n - C_n) = \frac{\lambda}{\sqrt{n}Z \log(U_{1,n})} - \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{\sqrt{n}(V_n(s) - s)}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds.$$

At this step, we need one version of the so-called hungarian construction. Precisely, Csörgö and al. (see [2]) have constructed a probability space holding a sequence of independent uniform random variables  $U_1, U_2, \dots$  and a sequence of brownian bridges  $B_1, B_2, \dots$  such that for each  $0 < \nu < 1/2$ , as  $n \rightarrow \infty$ ,

$$(4.12) \quad \sup_{1/n \leq s \leq 1-1/n} \frac{|\beta_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}).$$

where  $\{\beta_n(s) = \sqrt{n}(s - V_n(s)), 0 \leq s \leq 1\}$  is the uniform quantile process. From this and (4.11), we get

$$\begin{aligned} \sqrt{n}(I_n - C_n) &= \frac{\lambda}{\sqrt{n}Z \log(U_{1,n})} + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{\beta_n(s) - B_n(s)}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{B_n(s)}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds \\ &= \frac{\lambda}{\sqrt{n}Z \log(U_{1,n})} + \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{\beta_n(s) - B_n(s)}{s^{\frac{1}{2}-\nu}} * \frac{s^{\frac{1}{2}-\nu}}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds + \\ &\frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s) \left( \frac{1}{\zeta_n(s)(\log(\zeta_n(s)))^2} - \frac{1}{s(\log s)^2} \right) ds + \frac{\lambda}{Z} \int_{G(Z)}^{G_n(Z)} B_n(s) \left( \frac{1}{s(\log s)^2} \right) ds \\ (4.13) \quad &+ \frac{\lambda}{Z} \int_{1/n}^{G(Z)} B_n(s) \left( \frac{1}{s(\log s)^2} \right) ds =: R_{0,n} + R_{1,n} + R_{2,n} + R_{3,n} + N_n. \end{aligned}$$

We first claim that each of the  $R_{j,n}$  tends to zero in probability as  $n \rightarrow \infty$ .

Claim 1.1.  $R_{0,n} \rightarrow_p 0$ . We have

$$(4.14) \quad R_{0,n} = \frac{\lambda}{\sqrt{n}Z \log(U_{1,n})} = \frac{\lambda}{\sqrt{n}Z(\log(nU_{1,n}) - \log n)}$$

But it is well-known that  $\log(nU_{1,n}) = O_p(1)$  so that  $R_{0,n} = o_p(1)$ .

Claim 2.1.  $R_{1,n} \rightarrow_p 0$ . We have

$$(4.15) \quad R_{1,n} = \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \frac{\beta_n(s) - B_n(s)}{s^{\frac{1}{2}-v}} \frac{s^{\frac{1}{2}-v}}{\zeta_n(s)(\log(\zeta_n(s)))^2} ds$$

$$(4.16) \quad = O_p(n^{-v}) \int_{\frac{1}{n}}^{G_n(Z)} \frac{s^{-\frac{1}{2}-v}}{(\zeta_n(s)/s) \log \zeta_n(s)^2} ds =: O_p(n^{-v}) \times R'_{1,n}.$$

By Lemma 13 in [3],

$$\liminf_{\rho \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\rho^{-1} \leq \inf\{V_n(s)/s, 1/n \leq s \leq 1\} \leq \sup\{V_n(s)/s, 1/n \leq s \leq 1\} \leq \rho) = 1.$$

Recall again that  $G_n(Z) \rightarrow G(Z) > 0$ . Hence for any  $G(Z) > \varepsilon > 0$ , for large values of  $n$  and for large values of  $\rho$ , we have with probability at least greater than  $1-\varepsilon/3$  (denoted w.p. $\varepsilon/3$ ),

$$(4.17) \quad (\zeta_n(s)/s)^{-1} \leq \rho$$

and we have w.p. $\varepsilon/3$  for large  $n$ ,

$$(4.18) \quad 1/n \leq s \leq G_n(Z) \leq G(Z) + \varepsilon/2 \implies 0 \leq \zeta_n(s) \leq \max(V_n(s), s).$$

$$(4.19) \quad \leq \max(V_n(G(Z) + \varepsilon/2), G(Z) + \varepsilon/2)$$

and since  $V_n(G(Z) + \varepsilon/2) \rightarrow_p G(Z) + \varepsilon/2$ , we also get w.p. $\varepsilon/3$  for large  $n$ ,

$$(4.20) \quad 0 \leq V_n(G(Z) + \varepsilon/2) \leq G(Z) + \varepsilon$$

and then w.p. $2\varepsilon/3$ , for large  $n$

$$(4.21) \quad 1/n \leq s \leq G_n(Z) \leq G(Z) + \varepsilon \implies 0 \leq \zeta_n(s) \leq G(Z) + \varepsilon$$

This latter yields w.p. $2\varepsilon/3$ , for large  $n$

$$(4.22) \quad 1/n \leq s \leq G_n(Z) \leq G(Z) + \varepsilon \implies 0 \leq -1/\log \zeta_n(s) \leq -1/\log(G(Z) + \varepsilon)$$

By (4.17), (4.21), we arrive w.p. $\varepsilon$ , for large values of  $n$  and  $\rho$ ,

$$(4.23) \quad R'_{1,n} \leq \int_{\frac{1}{n}}^{G(Z)+\varepsilon} \frac{s^{-\frac{1}{2}-v}}{(\zeta_n(s)/s)(\log \zeta_n(s))^2} ds \leq \frac{\rho}{(\log(G(Z) + \varepsilon))^2} (G(Z) + \varepsilon)^{1/2-v}.$$

Finally  $R'_{1,n} = O_p(1)$  and then by (4.16),  $R_{1,n} \rightarrow_p 0$ .

Claim 3.1.  $R_{2,n} \rightarrow_p 0$ . We have

$$\begin{aligned}
(4.24) \quad R_{2,n} &= \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s) * \left( \frac{1}{\zeta_n(s)(\log(\zeta_n(s)))^2} - \frac{1}{s(\log s)^2} \right) ds \\
&= \frac{\lambda}{Z} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s) * \frac{(\pi_n(s) - s)(2 - \log(\pi_n(s)))}{\pi_n(s)^2 (\log \pi_n(s))^3} ds.
\end{aligned}$$

By (5) and (6) in [18], p.500, we have for any  $0 < c < \infty$ ,

$$(4.25) \quad \sup_{c/(n+1) < s < 1-c/(n+1)} \left| B_n(s) / \sqrt{s(1-s)} \right| = O_p(b_n)$$

and

$$(4.26) \quad \sup_{c/(n+1) < s < 1-c/(n+1)} \left| V_n(s) / \sqrt{s(1-s)} \right| = O_p(b_n)$$

with  $b_n = \sqrt{2 \log \log n}$ . This implies

$$\begin{aligned}
(4.27) \quad |R_{2,n}| &\leq \frac{\lambda}{Z\sqrt{n}} \int_{\frac{1}{n}}^{G_n(Z)} \frac{1-s}{s} \left| \frac{B_n(s)}{\sqrt{s(1-s)}} \right| * \frac{\left| \frac{\beta_n(s)}{\sqrt{s(1-s)}} \right| \times |1 - 2/\log \pi_n(s)|}{(\pi_n(s)/s)^2 \times |\log(\pi_n(s))|^2} ds \\
&= O_p(n^{-1/2} \log \log n) \int_{\min(\frac{1}{n}, G_n(Z))}^{\max(\frac{1}{n}, G_n(Z))} s^{-1}(1-s) * \frac{|1 - 2/\log \pi_n(s)|}{(\pi_n(s)/s)^2 \times |\log(\pi_n(s))|^2} ds
\end{aligned}$$

By the same methods that led to (4.17), we have, for any  $G(Z) > \varepsilon > 0$ , w.p. $\varepsilon$  for large values of  $n$  and  $\rho > 0$ ,

$$\begin{aligned}
(4.28) \quad R_{2,n} &\leq O_p(n^{-1/2} \log \log n) \rho^2 \frac{1 + 2/\log(G(Z) + \varepsilon)}{(\log(G(Z) + \varepsilon))^2} \int_{\frac{1}{n}}^{G(Z)+\varepsilon} s^{-1} ds \\
&= O_p(n^{-1/2} \log \log n) \rho^2 \frac{1 + 2/\log(G(Z) + \varepsilon)}{\log(G(Z) + \varepsilon)^2} (\log(G(Z)) + \log n) \rightarrow_p 0.
\end{aligned}$$

which finishes the proof of the claims.

Claim 4.1.  $R_{3,n} \rightarrow_p 0$ . Since  $G_n(Z) \rightarrow_p G(Z)$  as  $n \rightarrow \infty$ ,

$$\sup\{s(\log s)^2, s \in [\min(G_n(Z), G(Z)), \max(G_n(Z), G(Z))]\} \rightarrow_p G(Z)(\log(G(Z)))^2$$

and

$$\inf\{s(\log s)^2, s \in [\min(G_n(Z), G(Z)), \max(G_n(Z), G(Z))]\} \rightarrow_p G(Z)(\log(G(Z)))^2$$

and then

$$(4.29) \quad R_{3,n} = \frac{\lambda}{(Z \log Z)^2} (1 + o_p(1)) \int_{G(Z)}^{G_n(Z)} B_n(s) ds$$

But for any  $\varepsilon > 0$ , we have w.p. $\varepsilon$ , for large values of  $n$

$$(4.30) \quad \left| \int_{G(Z)}^{G_n(Z)} B_n(s) ds \right| \leq \int_{G(Z)-\varepsilon}^{G(Z)+\varepsilon} |B_n(s)| ds =: T_n(\varepsilon)$$

Now  $E(T_n(\varepsilon)) = E(\int_{G(Z)-\varepsilon}^{G(Z)+\varepsilon} |B_n(s)| ds) = (2\pi)^{-1} \int_{G(Z)-\varepsilon}^{G(Z)+\varepsilon} \sqrt{s(1-s)} = T(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . We conclude that for any  $\varepsilon$ , we get for large values of  $n$ , by Markov inequality

$$(4.31) \quad \forall \eta > 0, \mathbb{P}\left(\left|\int_{G(Z)}^{G_n(Z)} B_n(s) ds\right| \geq \eta\right) \leq E(T(\varepsilon))/\eta$$

that is

$$(4.32) \quad \forall \eta > 0, \limsup_{n \rightarrow \infty} \mathbb{P}\left(\left|\int_{G(Z)}^{G_n(Z)} B_n(s) ds\right| \geq \eta\right) \leq \mathbb{E}(T(\varepsilon))/\eta$$

Now, as  $\varepsilon \downarrow 0$ , we conclude that  $\int_{G(Z)}^{G_n(Z)} B_n(s) ds \xrightarrow{p} 0$  and thus  $R_{3,n} \xrightarrow{p} 0$  by (4.29) et (4.30).

Finally we achieve our proof by remarking that  $N_n$  is a riemannian integral of a brownian bridge and hence, is a gaussian and centered random variable with variance

$$(4.33) \quad \sigma_n^2 = \left(\frac{\lambda^2}{Z^2}\right) \int_{\frac{1}{n}}^{G(Z)} \int_{\frac{1}{n}}^{G(Z)} \frac{\min(s,t) - st}{s(\log s)^2 t(\log t)^2} ds dt$$

$$\frac{Z^2}{\lambda^2} \sigma_n^2 = \int_{\frac{1}{n}}^{G(Z)} \frac{1-s}{s(\log s)^2} \int_{\frac{1}{n}}^s \frac{dt}{(\log t)^2} ds + \int_{\frac{1}{n}}^{G(Z)} \frac{1}{(\log s)^2} \int_s^{G(Z)} \frac{(1-t)dt}{t(\log t)^2} ds$$

$$= \int_{\frac{1}{n}}^{G(Z)} \frac{1}{s(\log s)^2} \int_{\frac{1}{n}}^s \frac{dt}{(\log t)^2} ds + \int_{\frac{1}{n}}^{G(Z)} \frac{1}{(\log s)^2} \int_s^{G(Z)} \frac{dt}{t(\log t)^2} ds$$

$$(4.34) \quad - \int_{\frac{1}{n}}^{G(Z)} \int_{\frac{1}{n}}^{G(Z)} \frac{dt ds}{(\log t)^2 (\log s)^2},$$

Now by integrating by parts the first integral with  $s^{-1}(\log s)^{-2} = (-\int_s^{G(Z)} \frac{dt}{t(\log t)^2})'$ , we get

$$(4.35) \quad \frac{Z^2}{\lambda^2} \sigma_n^2 = 2 \int_{\frac{1}{n}}^{G(Z)} \frac{1}{(\log s)^2} \int_s^{G(Z)} \frac{dt}{t(\log t)^2} ds - \left(\int_{\frac{1}{n}}^{G(Z)} \frac{dt}{(\log t)^2}\right)^2.$$

Notice that  $-1/s(\log s)^2$  is the derivative of  $1/\log s$ . We conclude that

$$(4.36) \quad \frac{Z^2}{\lambda^2} \sigma_n^2 = \frac{-2}{\log G_n(Z)} \int_{1/n}^{G(Z)} (\log s)^{-2} ds + 2 \int_{1/n}^{G(Z)} (\log s)^{-3} - \left(\int_{\frac{1}{n}}^{G(Z)} \frac{dt}{(\log t)^2}\right)^2$$

$$(4.37) \quad \rightarrow \frac{Z^2}{\lambda^2} \sigma^2 = \frac{-2}{\log G(Z)} \int_0^{G(Z)} (\log s)^{-2} ds + 2 \int_0^{G(Z)} (\log s)^{-3} - \left(\int_0^{G(Z)} \frac{dt}{(\log t)^2}\right)^2,$$

with  $G(Z) = \exp(\lambda/(Z - y_0))$  for  $Z > y_0$ .

Proof of theorem2. We have for for  $\gamma > 0$  and  $x \geq 1$ ,  $F(x)=1-x^{-1/\gamma}$ , that is  $F^\leftarrow(s) = (1-s)^{-\gamma}$  for  $0 \leq s < 1$ . Thus by (4.7)

$$(4.38) \quad I_n = \frac{1}{nZ}q(Z - y_0) - \frac{1}{nZ}U_{1,n}^\gamma - \int_{\frac{1}{n}}^{G_n(Z)} V_n(s)^\gamma ds.$$

Put

$$(4.39) \quad D_n = \frac{1}{nZ}q(Z - y_0) - \frac{1}{Z} \int_{\frac{1}{n}}^{G_n(Z)} s^\gamma ds$$

By the mean value theorem, we get

$$(4.40) \quad I_n - D_n = \frac{1}{nZ}U_{1,n}^\gamma + \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} (V_n(s) - s)(\zeta_n(s))^{\gamma-1} ds,$$

where  $\zeta_n(s) \in [\min(s, V_n(s)); \max(s, V_n(s))]$  and

$$\begin{aligned} \sqrt{n}(I_n - D_n) &= \frac{1}{\sqrt{n}Z}U_{1,n}^\gamma - \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} \beta_n(s)(\zeta_n(s))^{\gamma-1} ds. \\ &= \frac{1}{\sqrt{n}Z}U_{1,n}^\gamma - \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} (\beta_n(s) - B_n(s)) (\zeta_n(s))^{\gamma-1} ds - \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G(Z)} B_n(s)((\zeta_n(s))^{\gamma-1} - s^{\gamma-1}) ds \end{aligned}$$

$$(4.41) \quad -\frac{\gamma}{Z} \int_{G(Z)}^{G_n(Z)} B_n(s)(s)^{\gamma-1} ds - \frac{\gamma}{Z} \int_{\frac{1}{n}}^{G(Z)} B_n(s)(s)^{\gamma-1} ds =: R_{0,n} + R_{1,n} + R_{2,n} + R_{3,n} + N_n$$

We first claim that each of the  $R_{j,n}$  tends to zero in probability as  $n \rightarrow \infty$ .

Claim 1.2.  $R_{0,n} \rightarrow_p 0$ . This immediate since  $\gamma > 0$  and  $U_{1,n} \rightarrow_p 0$  as  $n \rightarrow \infty$ .

Claim 2.2.  $R_{1,n} \rightarrow_p 0$ . Put  $0 < \nu < 1/2$  and  $\nu + 1/2 < 1 + \gamma > 1$  so that  $\delta = \frac{1}{2} + \nu - \gamma < 1$ . Then by (4.12),

$$(4.42) \quad \begin{aligned} \mathbf{R}_{1,n} &= O_p(n^{-\nu}) \int_{\frac{1}{n}}^{G_n(Z)} s^{\frac{1}{2}-\nu} \zeta_n(s)^{-1+\gamma} ds \\ &= O_p(n^{-\nu}) \int_{\frac{1}{n}}^{G_n(Z)} s^{-(\frac{1}{2}+\nu-\gamma)} (\zeta_n(s)/s)^{-1+\gamma} ds = O_p(n^{-\nu}) R'_{1,n}. \end{aligned}$$

We may use the same methods used in (4.23) to have for any  $\varepsilon > 0$ , for large values of  $n$  and  $\rho$ ,

$$\begin{aligned} 0 \leq R'_{1,n} &\leq \max(\rho^{-1+\gamma}, \rho^{1-\gamma}) \int_{\frac{1}{n}}^{G(Z)+\varepsilon} s^{-(\frac{1}{2}+\nu-\gamma)} ds. \\ 0 \leq R'_{1,n} &\leq \max(\rho^{-1+\gamma}, \rho^{1-\gamma}) \int_{\frac{1}{n}}^{G(Z)+\varepsilon} s^{-\delta} ds \end{aligned}$$

$$(4.43) \quad = \max(\rho^{-1+\gamma}, \rho^{1-\gamma})(G(Z) + \varepsilon)^{1-\delta}.$$

The formulae (4.42) and (4.43) together achieve the proof of Claim 2.2

Claim 3.2  $R_{2,n} \rightarrow_p 0$

$$(4.44) \quad R_{2,n} = -\frac{\gamma}{Z} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s) (\zeta_n(s)^{\gamma-1} - s^{\gamma-1}) ds$$

$$(4.45) \quad \mathbf{R}_{2,n} = -\frac{\gamma(\gamma-1)}{Z\sqrt{n}} \int_{\frac{1}{n}}^{G_n(Z)} B_n(s) \sqrt{n}(\zeta_n(s) - s) \pi_n(s)^{\gamma-2} ds$$

so that

$$(4.46) \quad |\mathbf{R}_{2,n}| \leq \frac{\gamma|\gamma-1|}{Z\sqrt{n}} \int_{\min(G_n(Z), 1/n)}^{\max(G_n(Z), 1/n)} |B_n(s)| \times |\sqrt{n}(V_n(s) - s)| \pi_n(s)^{\gamma-2} ds.$$

By using (4.25) and (4.26), we get

$$(4.47) \quad |\mathbf{R}_{2,n}| \leq O_p(n^{-1/2} \log \log n) \int_{\min(G_n(Z), 1/n)}^{\max(G_n(Z), 1/n)} s^{\gamma-1} (1-s) (\pi_n(s)/s)^{\gamma-2} ds.$$

As in what precedes, we have for any  $\varepsilon > 0$ , for large values of  $n$  and  $\rho > 0$ ,

$$(4.48) \quad |\mathbf{R}_{2,n}| \leq O_p(n^{-1/2} \log \log n) \max(\rho^{-2+\gamma}, \rho^{2-\gamma}) \int_0^{G(Z)+\varepsilon} s^{\gamma-1} ds.$$

Hence

$$(4.49) \quad |\mathbf{R}_{2,n}| \leq O_p(n^{-1/2} \log \log n) (G(Z) + \varepsilon)^\gamma.$$

This achieves the proof of Claim 3.2.

Claim 4.2.  $R_{3,n} \rightarrow_p 0$ . The proof is the same as that of Claim 4.1 with  $(-1/(s(\log s)^2))$  replaced by  $(s^{\gamma-2})$ .

It remains to remark as below that  $N_n$  is a centered gaussian random variable with variance

$$(4.50) \quad \theta_n^2 = \left(\frac{\gamma}{Z}\right)^2 \int_{\frac{1}{n}}^{G_n(Z)} \int_{\frac{1}{n}}^{G_n(Z)} (\min(s, t) - st) t^{\gamma-1} s^{\gamma-1} dt ds.$$

A direct calculation yields  $\theta_n^2 \rightarrow \theta^2 = (\gamma/Z)^2 G(Z)^{2\gamma+1} \left(\frac{2}{(\gamma+1)(2\gamma+1)} - \frac{G(Z)}{(\gamma+1)^2}\right)$  with  $G(Z) = ((Z - y_0)^{\frac{1}{\gamma}} \wedge 1)$ .

## 5. SIMULATIONS OF THE RESULTATS

We will simulate our results in the conditions of the senegalese data base ESAM (1996) related to a national survey with 3278 households. The income variable is available for ten (10) dsitrics with a minimal size of  $n=172$ . The poors headcounts in the differents areas vary from 35% to 81%. As a consequence, we make our simulations for the current estimations of  $\gamma$  (wich varies around 1,5) and  $\lambda$  (wich varies around 25000), with  $n=50$  and for several values of the headcounts (from 35% to 85%). For each case, we

will use  $B=100$  replications and report  $I_n$  the mean value of the empirical poverty intensity,  $D$  ou  $C$  the centering coefficient,  $E$  the mean quadratic error,  $ET^2$  the variance of the empirical intensity and the mean  $P$ -value for the normality test.

The following tables show that convergence are very fast even for a small size since the mean quadratic error is very small. The variances will permit to have the confidence intervals of the exact poverty intensity. The high values of the  $p$ -values  $P$  are very conclusive for the asymptotic normality.

### 5.1. Simulations for the poverty intensity for the Pareto law

$Q/n(\%)$	35	45	55	75	85
$\gamma = 0,8$	$I(\%) = 14,989$ $C = 15,29$ $E(\%) = 1,7$ $ET = 0,266$ $P(\%) = 26,38$	$I(\%) = 19,59$ $C = 19,79$ $E(\%) = 2,22$ $ET = 0,29$ $P(\%) = 20,49$	$I(\%) = 24,48$ $C = 24,10$ $E(\%) = 2,46$ $ET = 0,3$ $P(\%) = 24,94$	$I(\%) = 31,38$ $C = 31,14$ $E(\%) = 2,11$ $ET = 0,31$ $P(\%) = 28,25$	$I(\%) = 37,82$ $C = 37,47$ $E(\%) = 2,23$ $ET = 0,3$ $P(\%) = 27$
$\gamma = 2$	$I(\%) = 22,93$ $C = 22,95$ $E(\%) = 3,35$ $ET = 0,36$ $P(\%) = 23,89$	$I(\%) = 29,35$ $C = 29,79$ $E(\%) = 2,97$ $ET = 0,89$ $P(\%) = 24,62$	$I(\%) = 36,74$ $C = 36,28$ $E(\%) = 3,18$ $ET = 0,4$ $P(\%) = 23,5$	$I(\%) = 46,53$ $C = 46,92$ $E(\%) = 3,37$ $ET = 0,4$ $P(\%) = 23,88$	$I(\%) = 55,91$ $C = 55,93$ $E(\%) = 3,3$ $ET = 0,36$ $P(\%) = 20,84$

### 5.2. Simulations for the poverty intensity for the exponential law

$Q/n(\%)$	35	45	55	75	85
$\lambda = 2$	$I(\%) = 13,46$ $C = 13,6$ $E(\%) = 0,04$ $ET = 0,476$ $P(\%) = 25$	$I(\%) = 19,8$ $C = 19,73$ $E(\%) = 0,038$ $ET = 0,447$ $P(\%) = 25,18$	$I(\%) = 27,38$ $C = 27,55$ $E(\%) = 0,07$ $ET = 0,55$ $P(\%) = 24,26$	$I(\%) = 42,34$ $C = 42,4$ $E(\%) = 0,08$ $ET = 0,579$ $P(\%) = 22,88$	$I(\%) = 61,68$ $C = 0,48$ $E(\%) = 0,08$ $ET = 0,567$ $P(\%) = 24,74$
$\lambda = 25000$	$I(\%) = 13,70$ $C = 13,6$ $E(\%) = 0,04$ $ET = 0,476$ $P(\%) = 25$	$I(\%) = 19,8$ $C = 19,73$ $E(\%) = 0,03$ $ET = 0,517$ $P(\%) = 25,93$	$I(\%) = 27,39$ $C = 27,4$ $E(\%) = 0,06$ $ET = 0,55$ $P(\%) = 25,51$	$I(\%) = 42,57$ $C = 42,48$ $E(\%) = 0,07$ $ET = 0,574$ $P(\%) = 23,68$	$I(\%) = 61,83$ $C = 61,18$ $E(\%) = 0,07$ $ET = 0,563$ $P(\%) = 24,73$

## 6. CONCLUSION

We have been able to get the asymptotic normality of the poverty intensity for two special represents of the two extremal domains. The results are very well supported by the simulations. This opens the way for a general study of this poverty measure in the frame of extreme value theory first and futher for the nonweighted measures. Immediate steps after this study will consist of deep comments of the results as well as simulations from the 95 senegalease data.

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LERSTAD AND UNIVERSITÉ GASTON BERGER. BP 234, SAINT-LOUIS, SÉNÉGAL.

*E-mail address:* [gslo@ugb.sn](mailto:gslo@ugb.sn)

*Current address:* Lerstad and Ecole Normale Supérieure, Dakar, Sénégal.

*E-mail address:* [toubasall@yahoo.fr](mailto:toubasall@yahoo.fr)