Seasonal fractional ARIMA with stable innovations

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Abstract

We develop the theory of seasonaly fractionaly differenced ARIMA times series with stable infinite variance innovations establishing conditions for existence and invertibility. This is a finite parameter model which exhibits long range dependence, seasonality and high variability. We perform some simulations to illustrate the behavior of the model.

keywords : Seasonality, fractional ARIMA, Long memory, stable distributions.

AMS 2000 Subject Classification : 60E07, 62M10.

1 Introduction

Modelling times series using fractionally integrated processes with finite variance has been the purpose of many investigations. See Granger and Joyeux (1980) and Hosking (1981) who introduced the fractional *ARIMA* (p, d, q) model with finite variance innovations. Real words time series may display, in addition high

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variability. To take into account these two facts, the theory of the *SαS fractional ARIMA* (p, d, q) model were introduced and developed by Kokoska and Taqqu (1995).

However this model remains limited to described a large number of real word cyclical phenomenon with seasonal components exhibiting both long range dependence and high variability. See for instance Hassler (1994) or Porter-Hudak (1990) who examined monetary aggregates using a seasonal differenced process.

In this paper, we develop the theory of Seasonal Fractionally integrated ARIMA time series with stable innovations called *ARFSIMA-SαS* process. The Seasonal Fractionally integrated processes with stable innovations approach enables the modelling of many features of time series. Since it is a direct generalization of the Fractional ARIMA model of Kokoszka and Taqqu (1995), it allows to take into account three stylized facts: *long range dependence, seasonality and high variability* often encountered in financial data.

The use of linear processes with heavy tailed innovations (*high variability*) is not new and has received a great importance in modelling time series. Examples where such models appear to be appropriate have been found by Stuck and Kleiner (1974), who considered telephone signals, and Fama (1965), who modelled stock market prices.

The presence of *strong dependence* in times series was first highlighted in hydrological data. In economics, the long memory component was first detected in exchange rate data and in stock prices data.

In a number of papers, seasonal fluctuations have been found to play significant role in time series modelling. For inference and forecasting purposes, several procedures are used to treat this seasonality. See for instance Cunado *et al.* (2006) and the reference therein.

The remaining of this paper is structured as follows. In Section 2, we introduce this ARFISMA-SaS model. In Section 3, we establish conditions for existence and invertibility. Section 4 is devoted to the simulation study.

2 The model

A SαS random variable X has characteristic function

$$E\exp(iaX) = \exp\{-\sigma^{\alpha}|a|^{\alpha}\}, \qquad a \in \mathbb{R}, \quad 0 < \alpha \leq 2.$$

where i is the complex number such that $i^2 = -1$. The parameter σ is called the scale parameter, the parameter α is the index of stability. If $\alpha = 2$, X is gaussian with variance $2\sigma^2$. If $0 < \alpha < 2$, then $E|X|^p = \infty$ for $p \geq \alpha$.

Assume that $(Z_t)_{t\in\mathbb{Z}}$ is an independently and identically distributed *(i.i.d)* symmetric α-stable (SαS) random variables with $0 < \alpha \leq 2$. Let $\phi_p(B)$ and $\theta_q(B)$ be the well known non seasonal autoregressive and moving average polynomials with real coefficients defined by

$$\phi_p(B) = 1 - \phi_1 B - \ldots - \phi_p B^p;$$

$$\theta_q(B) = 1 + \theta_1 B + \ldots + \theta_q B^q.$$

A zero-mean process (X_t) is said a seasonal fractionally integrated process, denoted by $ARFISMA(p,\ d,\ q) \times (P,\ D,\ Q)_s$, if the following equation is satisfied

$$\phi_p(B)\Phi_P(B^s)(I-B)^d(I-B^s)^D X_t = \theta_q(B)\Theta_Q(B^s)Z_t, \tag{1}$$

where the long memory parameters d and D are fractional, B is the backward operator and s is the seasonal period that is the number of observations per period ($s = 1$ for annual data, $s = 2$ for half-yearly data, $s = 4$ for quarterly data, $s = 12$ for monthly data, $s = 52$ for weekly data). Here the polynomials $\Phi_P(B)$ and $\Theta_Q(B)$ are respectively the seasonal P order autoregressive polynomial and the seasonal Q order moving average polynomial.

The fractional difference operator $(1-B)^d$ is defined by its Mclaurin series

$$(1-B)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} B^j,$$

where $\Gamma(x)$ is the Euler gamma function defined by $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$.

The Seasonal Fractionally integrated processes with stable innovations approach enables the modeling of many features of financial market returns. Since it is a direct generalization of the FARIMA model of Granger and Joyeux (1980), it contains several extensions as:

1. The linear processes with Infinite variance studied by Fama (1965), Stuck and Kleiner (1974), or Brockwell and Davis (1991) when $D = d = 0$ and $P = Q = 0$.

2. The ARIMA processes with stable innovations introduced by Samorodnitsky and Taqqu (1994) when d is integer, $D = 0$ and $P = Q = 0$.

3. The fractional ARIMA processes with stable innovations of Kokoszka and Taqqu (1995) when $D = 0$ and $P = Q = 0$.

4. The SARIMA processes with white noise innovations of Brockwell and Davis (1991) when d and D are integers and $\alpha = 2$.

5. The seasonal fractionally integrated processes with finite variance ($\alpha = 2$) of Porter-Hudak (1990) and Hassler (1994) when $d = 0$ and $p = q = 0$.

In the present paper, we focus our attention on time series models with $P = Q = 0$ given by

$$\phi_p(B)(I-B)^d(I-B^s)^D X_t = \theta_q(B) Z_t \tag{2}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an *(i.i.d)* symmetric α-stable (SαS) random variables with $0 < \alpha \leq 2$.

The process specified by (2) is then a particular case of the seasonal fractionally integrated process defined by (1).

3 Existence and invertibility

According to Giraitis and Leipus (1995) or Reisen *et al.* (2006), one can easily show that

$$
\begin{aligned}
(I-B)^d(I-B^s)^D &= \prod_{j=0}^{[\frac{s}{2}]}\left[(1-e^{i\lambda_j}B)(1-e^{-i\lambda_j}B)\right]^{d_j} \\
&= \prod_{j=0}^{[\frac{s}{2}]}(1-2cos\lambda_j B+B^2)^{d_j}, \qquad (3)
\end{aligned}
$$

with $d_0=\frac{d+D}{2}$, $d_i=D$, $i=1,\ldots,[\frac{s}{2}]-1$, $d_{[\frac{s}{2}]}=\frac{D}{2}$, $\lambda_j=\frac{2\pi j}{s}$, $j=0,\ldots,[\frac{s}{2}]$. Thus the process defined by (2) can be rewritten as

$$
\phi_p(B)\prod_{j=0}^{[\frac{s}{2}]}(1-2cos\lambda_j B+B^2)^{d_j}X_t=\theta_q(B)Z_t.
$$

Let the polynomial $\phi_p(z)$ such that all its roots lie outside the unit circle. To show that Equation (2) has a unique causal moving average solution, we need to define the coefficients $(c_j)_{j\in\mathbb{Z}}$ by the following equation

$$
\frac{\theta_q(z)}{\phi_p(z)}\prod_{j=0}^{[\frac{s}{2}]}(1-2cos\lambda_j z+z^2)^{-d_j}=\sum_{j=0}^{\infty}c_j z^j. \qquad (4)
$$

It is easy to verify that

$$
c_j-\sum_{i=1}^{min(j,p)}\phi_i c_{j-i}=\psi_j(d,\nu)+\sum_{i=1}^{min(j,q)}\theta_i\psi_{j-i}(d,\nu), \qquad (5)
$$

where the ψ's are defined by

$$
\psi_j(d,\nu)=\sum_{\substack{0\le l_0,\cdots,l_{[\frac{s}{2}]}\le j,\\ l_0+\cdots+l_{[\frac{s}{2}]}=j}} C_{l_0}(d_0,\nu_0)\cdots C_{l_{[\frac{s}{2}]}}\left(d_{[\frac{s}{2}]},\nu_{[\frac{s}{2}]}\right). \qquad (6)
$$

The weigths $(C_l(d_i,\nu_i))_{l\in\mathbb{Z}}$ are the Gegenbauer polynomials and they can be computed using the following recursion formula:

$$
\begin{cases}
C_0(d_i,\nu_i)=1\\
C_1(d_i,\nu_i)=2d_i\nu_i\\
C_j(d_i,\nu_i)=2\nu_i\left(\frac{d_i-1}{j}+1\right)C_{j-1}(d_i,\nu_i)-\left(2\frac{d_i-1}{j}+1\right)C_{j-2}(d_i,\nu_i),\ \forall j>1.
\end{cases}
$$

See, for instance, Magnus *et al.* (1966) for more details on Gegenbauer polynomials.

Using expression (3) and according to Giraitis and Leipus (1995) or Woodward *et al.* (1998), we can determine the convergence region of the α order moment of the process given by (2) i.e conditions on the parameters d and D such that $\sum_{j=1}^{\infty} |c_j|^\alpha < 1$. The results are specified in the following theorem.

Theorem 1 *Under the following conditions* $|D+d| < 1 - \frac{1}{\alpha}$ *and* $|D| < 1 - \frac{1}{\alpha}$ *with* $1 < \alpha \leq 2$.

1. *If the polynomial* $\phi_p(z)$ *has no roots in the unit circle then the process defined by (2) has a unique causal moving average representation given by*

$$X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}, \tag{7}$$

where the coefficients $(c_j)_{j\in\mathbb{Z}}$ *are defined by (5).*

2. *If, in addition, the polynomial* $\theta_q(z)$ *has no roots in the unit circle then*

$$Z_n = \sum_{j=0}^{\infty} \tilde{c}_j X_{n-j} \qquad \textit{almost surely},$$

where the coefficients $(\tilde{c}_j)_{j\in\mathbb{Z}}$ *are defined by*

$$\tilde{c}_j = \pi_j(d,\nu) - \sum_{i=1}^{\min(j,p)} \phi_i \pi_{j-i}(d,\nu) + \sum_{i=1}^{\min(j,q)} \theta_i \tilde{c}_{j-i}. \tag{8}$$

The weights $(\pi_j(d,\nu))_{j\in\mathbb{Z}}$ *are such that* $\pi_j(d,\nu) = \psi_j(-d,\nu)$ *with the coefficients* $(\psi_j(-d,\nu))_{j\in\mathbb{Z}}$ *given in equation (6).*

Remark

According to Reisen *et al.* (2006), the model defined in (2) is an $ARUMA(p, \frac{d+D}{2}, D, \ldots, D, \frac{D}{2}, q)$ with gaussian innovations when $\alpha = 2$. Then, it is stationary and invertible if and only if $|D+d| < \frac{1}{2}$ and $|D| < \frac{1}{2}$ (see also, for instance, Giraitis and Leipus (1995).

Proof of Theorem 1:

The proof of Theorem 1 follows the great lines of the proof of Theorem 2.1. in Kokoszka and Taqqu (1995).

4 Simulation Study

In this section, we perform some simulations to illustrate the behavior of the model. We first describe a way to generated *ARFISMA*- $S\alpha S$ data and then examine features as long range dependence, seasonality and high variability in the simulated data.

Because there is no known technique to generate an exact ARFISMA in the stable case, we will approximate the infinite moving average (7) as follows, where M is finite

$$X_t = \sum_{j=0}^{M} c_j Z_{t-j}, \qquad t = 1, \ldots, N$$

where the non random constants c_j are defined by (5).

We simulate a *ARFISMA*- $S\alpha S$.

$$(I - B)^d (I - B^s)^D (X_t - \mu) = Z_t \tag{9}$$

where $d = 0.15$, $D = 0.20$, $\mu = 0$ is the mean of the process and (Z_t) is a SαS-stable process. We consider sample sizes $n = 1000$ and seasonal periods $s = 2$, $s = 4$ and $s = 6$. The values of the long memory parameters are chosen such that $|D| < 1 - \frac{1}{\alpha}$ and $|d + D| < 1 - \frac{1}{\alpha}$ according to the assumptions of Theorem 1. Thus the model defined by (9) has a long memory seasonal component at frequencies $\lambda_j = \frac{2\pi j}{s}$ for $j = 1, 2, \ldots \left[\frac{s}{2}\right]$.

4.1 Gaussian case

In this subsection, we suppose that the driving noise (Z_t) of the model (9) has a gaussian distribution with zero mean and its variance is taken to be one. We plotted on Figures 1(left)-3 (left) the trajectory of the process (X_t) when the seasonal periods are respectively equal to $s = 2$, $s = 4$ and $s = 6$. In these three graphs, we observe that the underlying processes seem to be weakly stationary.

We turn now to the study of the long range dependence of the process (X_t) defined by (9). The well known tools usually employed to detect presence of long memory are the covariance structure and the spectral approach. The spectral density of Gaussian *ARFISMA* model has been widely studied in the literature. The spectral density of model (9) is given by

$$f(\lambda) = \frac{\sigma_Z^2}{2\pi}\left[2\sin(\lambda s/2)\right]^{-2D}\left[2\sin(\lambda/2)\right]^{-2d} \tag{10}$$

for $-\pi \leq \lambda \leq \pi$. This density is unbounded at the seasonal frequencies $\lambda_j = \frac{2\pi j}{s}$ for $j = 1, 2, \ldots \left[\frac{s}{2}\right]$. See for instance [14]. In finite samples, the spectral density is approximated by the normalized periodogram defined as follows

$$\tilde{I}_n(\lambda) = \left(\sum_{t=1}^{n} X_t^2\right)^{-1}\left|\sum_{t=1}^{n} X_t e^{-it\lambda}\right|^2. \tag{11}$$

Figure 1, Figure 2 and Figure 3 display on the right the sample normalized periodogram given by (11) applied to simulated data with respectively $s = 2$, $s = 4$ and $s = 6$. The three graphs show picks at the seasonal frequencies. In the spectral domain, a peak in the spectral density at a given frequency λ indicates a cycle of period $\frac{2\pi}{\lambda}$ in the process.

In the classical case where the variance of the marginal distribution is finite and correlation exist, the sample autocorrelation function (ACF) of the data $\hat{\rho}(h)$ defined by

$$\hat{\rho}(h) = \frac{\sum_{i=1}^{n-|h|}(X_i - \bar{X})(X_{i+h} - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \tag{12}$$

estimates the theoretical autocorrelation function $\rho(h)$. The bottom graphs of Figure 1, Figure 2 and Figure 3 illustrate the behavior of (12) when $s = 2$, $s = 4$ and $s = 6$.

4.2 Heavy tailed case

We now abstract from the gaussian case and consider the stable case when the index of stability $\alpha \neq 2$. In the sequel, we set $\alpha = 1.7$. The time series plot showing evidence of heavy tails (great variability) are shown in the left graphs of Figures 4, 5 and 6. We observe a great variability in the variance of the process.

When heavy tails are present as noted above, the variances are infinite and the following heavy tailed modification of the sample autocorrelation function is more appropriate :

$$\hat{\rho}_H(h) = \frac{\sum_{i=1}^{n-|h|} X_i X_{i+h}}{\sum_{i=1}^{n} X_i^2}.$$

Looking at the bottom graphs of Figures 4, 5 and 6, we see that in finite sample the speed of convergence of $\hat{\rho}_H(h)$ towards 0 is very slow as $h \to \infty$.

In the stable setting, where variances are infinite, periodograms and spectra do not exist. Nevertheless, Mikosch *et al.* (1995) reported that there is no intrinsic problem involved in defining the sample periodogram from the data in the usual fashion.

The right panels of Figures 4, 5 and 6 show the behavior of the sample normalized periodogram of the process with picks located at the seasonal frequencies $\lambda_j = \frac{2\pi j}{s}$, $j = 1, 2, \ldots, s$.

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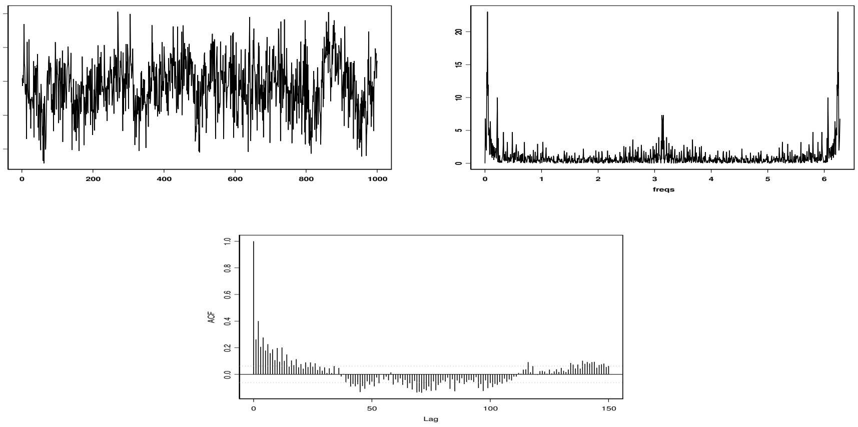

Figure 1: Trajectory (top left), periodogram (top right), acf (bottom) for $s = 2$, $\alpha = 2$,

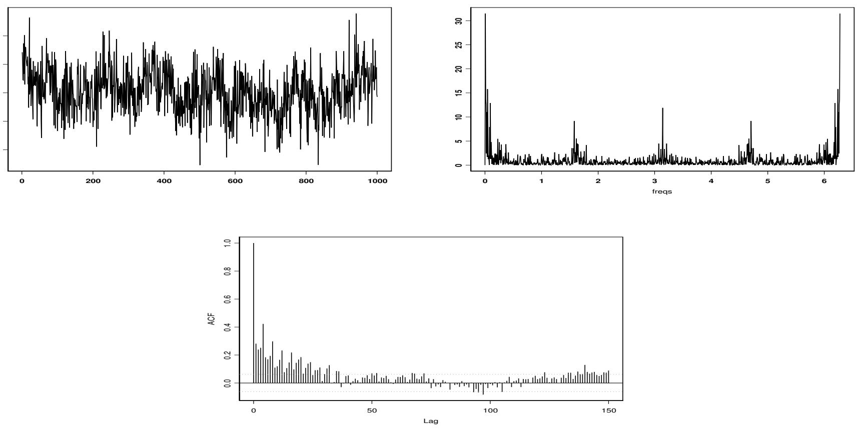

Figure 2: Trajectory (Top left), periodogram (Top right), acf (Bottom), for $s = 4$, $\alpha = 2$

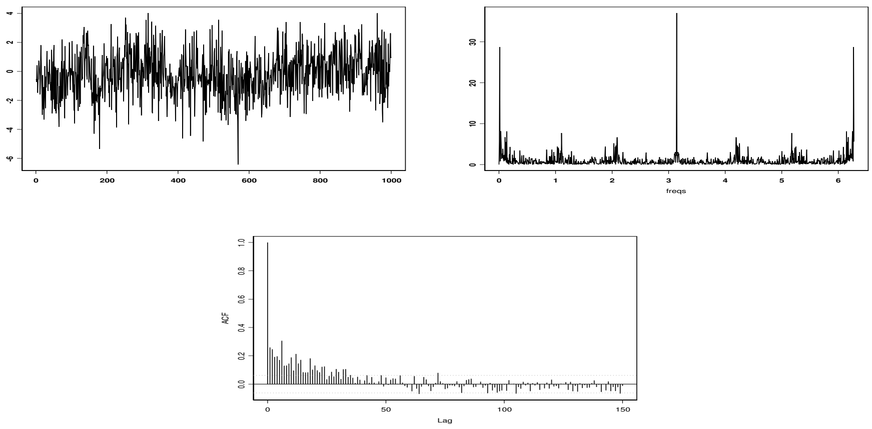

Figure 3: Trajectory (Top left), periodogram (Top right), acf (Bottom), for $s = 6$, $\alpha = 2$

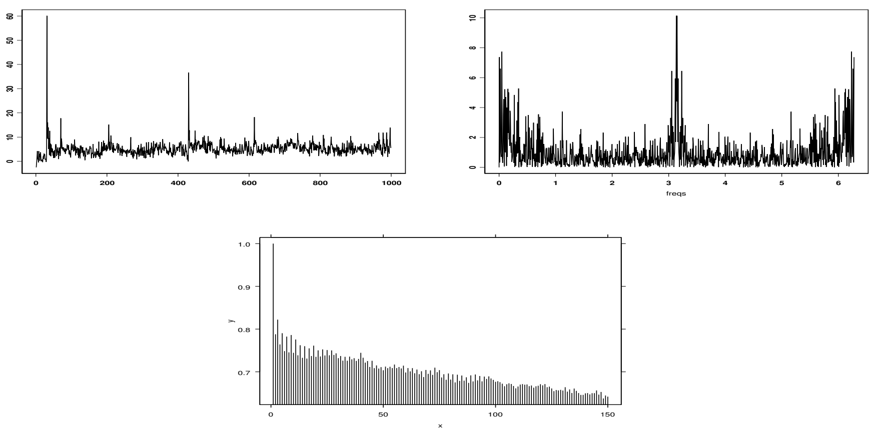

Figure 4: Trajectory (Top left), periodogram (Top right), acf heavy tailed (bottom) for $s = 2$, $\alpha = 1.7$

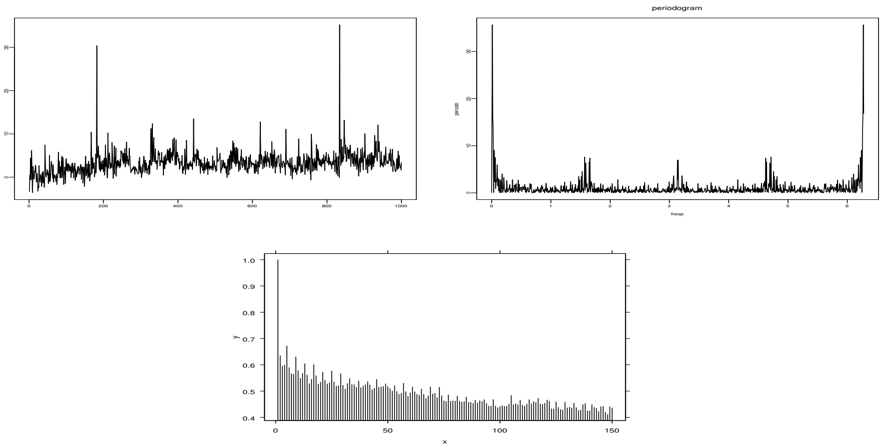

Figure 5: Trajectory (Top left), periodogram (Top right), acf heavy tailed (bottom) for $s = 4$, $\alpha = 1.7$

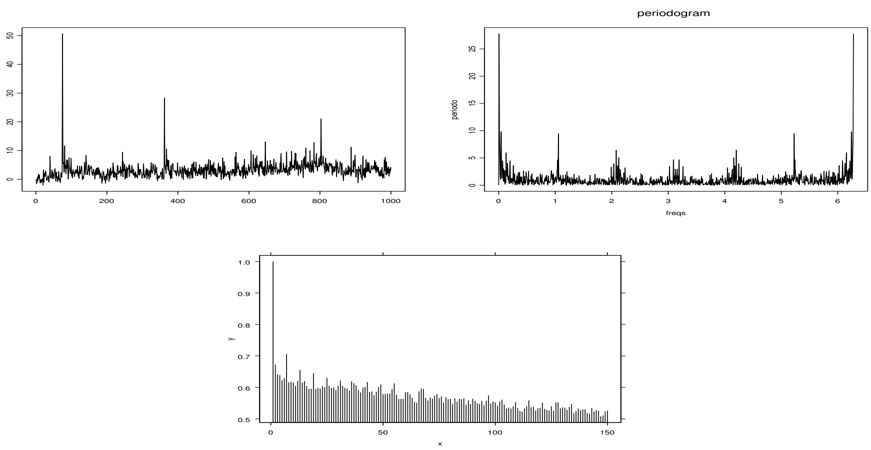

Figure 6: Trajectory (Top left), periodogram (Top right), acf heavy tailed (bottom) for $s = 6$, $\alpha = 1.7$