Tail behavior of threshold models with innovations in the domain of attraction of the double exponential distribution

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Abstract

We consider a two-regime threshold autoregressive model where the driving noises are sequences of iid random variables with common distribution function $F_i,\ i = 1, 2$ which belongs to the domain of attraction of double exponential distribution. If in addition, $F_i \in S_r(\gamma)$ i.e $\lim_{x\to\infty} \frac{\bar{F}_i(x-y)}{\bar{F}_i(x)} = e^{\gamma y}$, for each $y \in \mathbb{R}$, and $\lim_{x\to\infty} \frac{\overline{F_i * F_i}(x)}{\bar{F}_i(x)} =: d_i < \infty$ where $F * G$ denote the convolution of the distribution function and $\overline{F} = 1 - F$, we determine the tail behavior of the process and give the exact values of the coefficient.

Key Words: Tail behavior; Convolution tails; Stochastic Volatility Model; Threshold Autoregressive Model.

1 Introduction

A distribution function F is in the domain of attraction of the extreme value distribution $\Lambda(x) = \exp\{-e^{-x}\}$, $x \in \mathbb{R}$ if there exists $a_n > 0$, $b_n \in \mathbb{R}$ $(n \geq 1)$ such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x), \quad x \in \mathbb{R}. \tag{1}$$

A distribution F is in the class $S_r(\gamma)$ for $\gamma \geq 0$ if $F(x) < 1$ for all x and

$$\lim_{x\to\infty} \frac{\bar{F}(x-y)}{\bar{F}(x)} = e^{\gamma y}, \text{for each } y \in \mathbb{R}, \tag{2}$$

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and

$$\lim_{x\to\infty} \frac{\overline{F*F}(x)}{\bar{F}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}\left(X_1+X_2>x\right)}{\mathbb{P}\left(X_1>x\right)} =: d < \infty. \tag{3}$$

where $\bar{F} = 1 - F$ and $F * G$ denote the convolution of the distribution functions F and G. The constant d is known to equal $2\mathbb{E}e^{\gamma X_1}$. In the sequel, we set $m_F(\gamma) = \mathbb{E}e^{\gamma X}$ when F is the distribution function of the random variable X. See Cline (1987) and Embrechts (1985) for further details on convolution tails.

Switching regimes is stylized facts encountered in financial data analysis, concerning either financial returns, interest rates or volatilities. The threshold autoregressive (TAR) model was introduced by Tong (1978) and has since become quite popular in non-linear time series modelling. The TAR model can be seen as a stochastic difference equation. The tail behavior of a stationary solution of such equation has been widely studied in a variety of context. See for example Kesten (1973) and Resnick and Willekens (1991). In our framework, the TAR model is a stochastic difference equation where the multiplicative coefficient and the noise term are dependent. To our knowledge the litterature is not abundant for this framework. Diop and Guégan (2004) studied the threshold autoregressive stochastic volatility model where the driving noises are sequences of iid regularly varying random variables. The aim of this paper is to study the tail behavior of a two-regime threshold autoregressive model when the driving moise of each regime has distribution function $F_i \in D(\Lambda) \cap S_r(\gamma),\ i = 1, 2$. Precisely we determine the exact value of the coefficient in the tail behavior of the stationary solution when the model is stationary in some regimes and mildly explosive in others.

The rest of this paper is organized as follows: Section 2 describes the model and conditions for strict stationarity are provided. Some preliminary results with respect to the innovation processes are given. Section 3 presents the main results.

2 The model

The threshold autoregressive (TAR) model is defined by the following relation

$$\alpha_t = \begin{cases} \phi_1 \alpha_{t-1} + Z_t^{(1)}, & \text{if } Y_{t-1} \le \tau, \\ \phi_2 \alpha_{t-1} + Z_t^{(2)}, & \text{if } Y_{t-1} > \tau, \end{cases} \tag{4}$$

where τ and ϕ_i are non random constants and with threshold variable Y_{t-1}.

2.1 Assumptions

We will use the following assumptions.
H_1 - $(Z_1^{(i)})$ is sequence of iid random variables $(i = 1, 2)$ and satisfied the following conditions:

$$\mathbb{E}[\log^+ Z_1^{(i)}] < +\infty, \tag{5}$$

where $\log^+ x = max(0, \log x)$.
H_2 - For each $i = 1, 2$, the two sequences of random variables $(Z_t^{(i)})_t$ and $(Y_t)_t$ are independent and $(Z_t^{(1)})_t$ and $(Z_t^{(2)})_t$ are independent.
H_3 - The sequence of iid random variables $(Z_t^{(i)})_t$ whose common distribution F_i is both in the domain of attraction of $\Lambda(x) = \exp\{-e^{-x}\}, x \in \mathbb{R}$ and in $S_r(\gamma)$, $\gamma \geq 0$ and satisfies the tail balancing condition

$$\lim_{x\to\infty} \frac{\mathbb{P}(Z_1^{(i)} > x)}{\mathbb{P}(|Z_1^{(i)}| > x)} = p, \quad \lim_{x\to\infty} \frac{\mathbb{P}(Z_1^{(i)} < -x)}{\mathbb{P}(|Z_1^{(i)}| > x)} = 1 - p. \tag{6}$$

We define $q = \mathbb{P}(Y_t \leq \tau)$ and $I_{1t} = \mathbf{1}_{\{Y_{t-1} \leq 0\}}$, $I_{2t} = 1 - I_{1t}$. Then equation (4) can be rewritten as:

$$\alpha_t = \phi_{(t)}\alpha_{t-1} + Z_t, \tag{7}$$

where

$$\phi_{(t)} = \phi_1 I_{1t} + \phi_2 I_{2t} \qquad \text{and} \qquad Z_t = Z_t^{(1)} I_{1t} + Z_t^{(2)} I_{2t}.$$

We easily check that the tail balancing condition (6) holds for random variables (Z_k) whose distribution function $F = qF_1 + (1-q)F_2$.

2.2 Preliminary result

The equation (7) is a stochastic difference equation where the pairs $(\phi_t, Z_t)_t$ are sequences of iid $\mathbb{R}^2$-valued random variables under H_1-H_2. The next proposition gives the strict stationarity of the process $(\alpha_t)_t$ defined in (7). The result follows from Theorem 1 of Brandt (1986).

Proposition 1 (strict stationarity) *Assume H_1 and H_2 and suppose that* $| \phi_1 |^q | \phi_2 |^{1-q} < 1$. *Then, for all $t \in \mathbb{Z}$, the series $(\alpha_t)_t$ defined in (7) admits the following expansion*

$$\alpha_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j-1} \phi_{t-k} \right) Z_{t-j}, \tag{8}$$

which converges almost surely. Then the process $(\alpha_t)_t$ is the unique strictly stationary solution of (7).

Proposition 2 *Let* $F = qF_1 + (1-q)F_2$.
(i) If $F_i \in D(\Lambda), \quad i = 1,2$ *then* $F \in D(\Lambda)$.
(ii) If $F_i \in S_r(\gamma), \quad i = 1,2,\ \gamma \geq 0$ *then* $F \in S_r(\gamma)$.

Proof :
Without loss of generality, we can assume that $\dfrac{\bar{F}_2(x)}{\bar{F}_1(x)}$ tends to some constant $c \geq 0$ as $x \to \infty$ which we denote by $\bar{F}_2 \sim c\bar{F}_1$ with $c \geq 0$. Then the proof of (i) follows from Proposition 3.3.28 of Embrechts *et al.* (1997). Indeed

$$\frac{\bar{F}(x)}{\bar{F}_1(x)} = q + (1-q)\frac{\bar{F}_2(x)}{\bar{F}_1(x)}$$

which tends to some positive constant $k > 0$ as $x \to \infty$. Hence F belongs to $D(\Lambda)$.
Now we prove *(ii)*.
First, using $\bar{F}_2 \sim c\bar{F}_1$ with $c \geq 0$, it is easy to show by simple calculations that $\dfrac{\bar{F}(x-y)}{\bar{F}(x)} \longrightarrow \quad e^{\gamma y}$ as $x \to \infty$.
It suffices now to show that $\lim\limits_{x\to\infty} \dfrac{\overline{F*F}(x)}{\bar{F}(x)} = d < \infty$.
Now, we use the decomposition

$$\begin{aligned}\mathbb{P}(Z_1 + Z_2 > x) &= q^2\mathbb{P}(Z_1^{(1)} + Z_2^{(1)} > x) + q(1-q)\mathbb{P}(Z_1^{(1)} + Z_2^{(2)} > x) \\ &+ q(1-q)\mathbb{P}(Z_1^{(2)} + Z_2^{(2)} > x) + (1-q)^2\mathbb{P}(Z_1^{(2)} + Z_2^{(2)} > x).\end{aligned}$$

Then

$$\begin{aligned}\frac{\mathbb{P}(Z_1 + Z_2 > x)}{\mathbb{P}(Z_1 > x)} &= \frac{q^2\mathbb{P}(Z_1^{(1)} + Z_2^{(1)} > x)}{\mathbb{P}(I_{11}Z_1^{(1)} + I_{21}Z_1^{(2)} > x)} + \frac{q(1-q)\mathbb{P}(Z_1^{(1)} + Z_2^{(2)} > x)}{\mathbb{P}(I_{11}Z_1^{(1)} + I_{21}Z_1^{(2)} > x)} \\ &+ \frac{q(1-q)\mathbb{P}(Z_1^{(2)} + Z_2^{(1)} > x)}{\mathbb{P}(I_{11}Z_1^{(1)} + I_{21}Z_1^{(2)} > x)} + \frac{(1-q)^2\mathbb{P}(Z_1^{(2)} + Z_2^{(2)} > x)}{\mathbb{P}(I_{11}Z_1^{(1)} + I_{21}Z_1^{(2)} > x)}.\end{aligned}$$

We can write

$$\frac{\overline{F*F}(x)}{\bar{F}(x)} = \frac{q^2\overline{F_1*F_1}(x)}{\bar{F}(x)} + \frac{2q(1-q)\overline{F_1*F_2}(x)}{\bar{F}(x)} + \frac{(1-q)^2\overline{F_2*F_2}(x)}{\bar{F}(x)}, \qquad (9)$$

where $\bar{F}(x) = q\bar{F}_1(x) + (1-q)\bar{F}_2(x)$.
Since F_1 and F_2 belong to the class $S_r(\gamma)$, we have

$$\lim_{x\to\infty} \frac{\overline{F_1*F_1}(x)}{\bar{F}_1(x)} = d_1 < \infty$$

and

$$\lim_{x\to\infty}\frac{\overline{F_2 * F_2}(x)}{\bar{F}_2(x)} = d_2 < \infty.$$

Then

$$\frac{q^2\overline{F_1 * F_1}(x)}{\bar{F}(x)} \to \frac{q^2 d_1}{q + c(1-q)}.$$

Similarly, we have

$$\frac{(1-q)^2\overline{F_2 * F_2}(x)}{\bar{F}(x)} \to \frac{(1-q)^2 c d_2}{q + c(1-q)}.$$

Using Theorem 1 of Cline (1986), we show that

$$\frac{q(1-q)\overline{F_1 * F_2}(x)}{\bar{F}(x)} \to \frac{q(1-q)(d_1 + c d_2)}{q + c(1-q)}.$$

The proposition is entirely demonstrated.

3 Main Results

Our aim in this section is to establish the tail behavior of the stationary distribution of $(\alpha_t)_t$ defined in (4).

Theorem 1 *Let $(\alpha_t)_t$ be the stationary solution of equation (7) and the process $(Z_t)_t$ be an iid sequence of random variables with common distribution $F \in D(\Lambda) \cap S_r(\gamma)$ satisfying (6). Suppose that the assumptions of Proposition 1 hold. Then the tail behavior of the stationary distribution of $(\alpha_t)_t$ is*

$$\lim_{x\to\infty}\frac{\mathbb{P}(\alpha_t > x)}{\bar{F}(x)} = \left(\sum_{i=0}^{n}\beta_i m_{G_0 * G_1 * \ldots * G_{i-1}}(\gamma) m_{G_{i+1}}(\gamma)\ldots m_{G_n}(\gamma)\right)\left(\sum_{i=0}^{n+1} m_{\alpha_t}\left(\gamma\phi_1^i\phi_2^{n+1-i}\right)p_{n+1,i}\right), \tag{10}$$

where

$$m_{G_j}(\gamma) = \sum_{k=0}^{j} m_F\left(\gamma\phi_1^k\phi_2^{j-k}\right)p_{j,k} \qquad p_{j,k} = \mathcal{C}_j^k q^k (1-q)^{j-k}$$

$m_{G_0 * G_1 * \ldots * G_i}$ *can be computed using the following recursion formula :*

$$\left\{\begin{array}{rcl} m_{G_0 * G_1}(\gamma) &=& q m_{F_1}(\gamma) m_F(\gamma\phi_1) + (1-q) m_{F_2}(\gamma) m_F(\gamma\phi_2), \\ m_{G_0 * G_1 * \ldots * G_{i+1}}(\gamma) &=& q m_{F_1}(\gamma) m_{G_0 * G_1 * \ldots * G_i}(\gamma) + (1-q) m_{F_2}(\gamma) m_{G_0 * G_1 * \ldots * G_i}(\gamma) \end{array}\right. \tag{11}$$

and

$$\beta_k = \begin{cases} q^k & \text{if} \quad \phi_1 = 1, \quad |\phi_2| < 1, \\ p^{-1}q^k(p\delta_k + (1-p)\delta_{k+1}) & \text{if} \quad \phi_1 = -1, \quad |\phi_2| < 1, \\ (1-q)^k & \text{if} \quad \phi_2 = 1, \quad |\phi_1| < 1, \\ p^{-1}(1-q)^k(p\delta_k + (1-p)\delta_{k+1}) & \text{if} \quad \phi_2 = -1, \quad |\phi_1| < 1, \\ 0 & \text{if} \quad |\phi_1| < 1, \quad |\phi_2| < 1, \end{cases} \tag{12}$$

with

$$\delta_k = \begin{cases} 1 & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd.} \end{cases} \tag{13}$$

Remark : We may give an financial example of model (4) introduced by Breidt (1996) for a financial return Y_t defined by the following relation

$$Y_t = \sigma \exp(\frac{\alpha_t}{2})\epsilon_t, \tag{14}$$

where α_t is an open-loop threshold autoregressive process (Tong 1990, p. 101)

$$\alpha_t = \begin{cases} \phi_1 \alpha_{t-1} + Z_t^{(1)}, & \text{if}\, Y_{t-1} \leq 0, \\ \phi_2 \alpha_{t-1} + Z_t^{(2)}, & \text{if}\, Y_{t-1} > 0. \end{cases} \tag{15}$$

This model is called a threshold autoregressive stochastic volatility model (TARSV). The log-volatility process (α_t) has a piecewise linear structure. It switches between two first-order autoregressive process according to the sign of the previous return. In this framework, σ is positive constant and $(\epsilon_t)_t$ is a sequence of independent and identically distributed random variables with zero mean and its variance is taken to be one. When either $|\phi_1| = 1$ and $|\phi_2| \neq 1$ or $|\phi_2| = 1$ and $|\phi_1| \neq 1$, the process defined in (15) is stationary in some regimes and mildly explosive in others. These models are stationary in some regimes and mildly explosive in others. See Gongalo and Montesinos (2002). Gouriéroux and Robert (2006) studied the ACR(1) process where there is a switching between white noise and a random walk.

Before proving Theorem 1 we establish three lemmas. The next lemma is due to Davis and Resnick (1988). Its proof will then be omitted. The second lemma is an extension of Proposition 1.2 in Davis and Resnick (1988) where the hypothesis of independence is relaxed. See also Cline (1986). They are needed for the proof of the tail behavior of (α_t).

Lemma 1 *Suppose $F \in D(\Lambda)$ so the Balkema and de Haan (1972) representation holds*

$$\bar{F}(x) = \theta(x) \exp\left\{-\int_{z_0}^{x} \frac{1}{f(t)} dt\right\}, \tag{16}$$

for some z_0 and $x > z_0$ where $\theta(x) \to \theta \in (0, \infty)$ as $x \to \infty$,
$f > 0$ is absolutely continuous on $(z_0, \infty)$with density f' and $\lim_{u\to\infty} f'(u) = 0$.
Given $\epsilon > 0$, there exists $x_0 = x_0(\varepsilon)$ such that for $x \geq x_0$

$$\frac{\overline{F}(c^{-1}x)}{\overline{F}(x)} \leq (1+\varepsilon)\left(\frac{f(x)}{x}\right)^{1/\varepsilon}\left(\frac{c}{\varepsilon(1-c)}\right)^{1/\varepsilon}, \tag{17}$$

for any $0 < c < 1$.

This following lemma is quite general since it does not require the hypothesis of independence between the Y_i's.

Lemma 2 *Let $\{Y_i,\ 1 \leq i \leq n\}$ be random variables with distribution function G_i and suppose $F \in S_r(\gamma)$. If*

$$\lim_{x\to\infty} \frac{\mathbb{P}(Y_i > x)}{\overline{F}(x)} = \alpha_i \in [0, \infty] \tag{18}$$

for $i = 1, ..., n$ then for all $n \geq 2$,

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\sum_{i=1}^{n} Y_i > x\right)}{\overline{F}(x)} = \sum_{i=1}^{n} \alpha_i m_{G_1 * G_2 * \ldots * G_{i-1}}(\gamma) m_{G_{i+1}}(\gamma) \ldots m_{G_n}(\gamma). \tag{19}$$

Proof :
When $n = 2$, then this lemma is the formulation of Theorem 1 of Cline (1986). When $n > 2$, set $S_n = \sum_{i=1}^{n} Y_i$ and G its distribution function. Applying (19), when $n = 2$, we have

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\sum_{i=1}^{n+1} Y_i > x\right)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{\mathbb{P}\left(S_n + Y_{n+1} > x\right)}{\overline{F}(x)}$$

$$\begin{aligned} &= \alpha_{n+1} m_G(\gamma) + \left(\sum_{i=1}^{n} \alpha_i m_{G_1 * G_2 * \ldots * G_{i-1}}(\gamma) m_{G_{i+1}}(\gamma) \ldots m_{G_n}(\gamma)\right) m_{G_{n+1}}(\gamma) \\ &= \sum_{i=1}^{n+1} \alpha_i m_{G_1 * G_2 * \ldots * G_{i-1}}(\gamma) m_{G_{i+1}}(\gamma) \ldots m_{G_{n+1}}(\gamma). \end{aligned} \tag{20}$$

Lemma 3

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(\sum_{j=n+1}^{\infty}\left(\prod_{k=0}^{j-1}\phi_{(t-k)}\right)Z_{t-j} > x\right)}{\mathbb{P}(Z_t > x)} = 0. \tag{21}$$

Proof: For $\varepsilon > 0$ and for all b such that $1 < b^{1/\varepsilon} < \frac{1}{E|\phi_{(1)}|^{1/\varepsilon}}$, we have

$$\mathbb{P}\left(\sum_{j=n+1}^{\infty}\left(\prod_{k=0}^{j-1}\phi_{(t-k)}\right)Z_{t-j} > x\right) \leq \sum_{j=n+1}^{\infty}\sum_{k=0}^{j}\mathbb{P}\left\{\mid \phi_1^k\phi_2^{j-k} \mid\mid Z_{t-j} \mid > b^{n-j}(b-1)x\right\} p_{j,k},$$

where $p_{j,k} = \mathcal{C}_j^k q^k(1-q)^{j-k}$.
Using the tail balancing condition (6), we have

$$\sum_{k=0}^{j}\frac{\mathbb{P}\left(\mid Z_{t-j} \mid > x\delta_k\right)}{\mathbb{P}\left(Z_{t-j} > x\delta_k\right)}\frac{\mathbb{P}\left(Z_{t-j} > x\delta_k\right)}{\mathbb{P}(Z_t > x)}p_{j,k} = \sum_{k=0}^{j}\frac{\overline{F}(x\delta_k)}{\overline{F}(x)}p^{-1}p_{j,k},$$

where $\delta_k = b^{n-j}(b-1)\mid \phi_1^k\phi_2^{j-k}\mid^{-1}$. Since $b > 1$, it is readily seen that $\delta_k^{-1} < 1$ for all $j > n$ and $j > k$.
Hence applying Lemma 1 with $c = \delta_k^{-1}$, given $\varepsilon > 0$, there exists x_0 such that for $x \geq x_0$, we have

$$\sum_{k=0}^{j}\frac{\overline{F}(x\delta_k)}{\overline{F}(x)}p_{j,k} \leq \sum_{k=0}^{j}(1+\varepsilon)\left(\frac{f(x)}{x}\right)^{1/\varepsilon}\left(\frac{\delta_k^{-1}}{\varepsilon(1-\delta_k^{-1})}\right)^{1/\varepsilon}p_{j,k}.$$

Now following the proof of Proposition 1.1 of Davis and Resnick (1988), we give an upper bound of

$$K\left(\frac{f(x)}{x}\right)^{1/\varepsilon}\sum_{j=n+1}^{\infty}\sum_{k=0}^{j}\left[\left(b^{n-j}(b-1)\right)^{-1}\left(\mid \phi_1^k\phi_2^{j-k}\mid\right)\right]^{1/\varepsilon}p_{j,k}.$$

Observe that

$$\sum_{k=0}^{j}\left[\left(b^{n-j}(b-1)\right)^{-1}\left(\mid \phi_1^k\phi_2^{j-k}\mid\right)\right]^{1/\varepsilon}p_{j,k} = \left(b^n(b-1)\right)^{-1/\varepsilon}\left(\mathbb{E}\mid b\phi_{(1)}\mid^{1/\varepsilon}\right)^j.$$

This implies

$$\begin{aligned}\mathbb{P}\left(\sum_{j=n+1}^{\infty}\left(\prod_{k=0}^{j-1}\phi_{(t-k)}\right)Z_{t-j} > x\right) &\leq \sum_{j=n+1}^{\infty}\sum_{k=0}^{j}\mathbb{P}\left\{\mid \phi_1^k\phi_2^{(j-k)}\mid\mid Z_{t-j}\mid > b^{n-j}(b-1)x\right\} \\ &\leq K\left(\frac{f(x)}{x}\right)^{1/\varepsilon}\sum_{j=n+1}^{\infty}\left(\mathbb{E}\mid b\phi_{(1)}\mid^{1/\varepsilon}\right)^j.\end{aligned}$$

Since $\mathbb{E} \mid b\phi_{(1)} \mid^{1/\varepsilon} < 1$ and $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f'(x) = 0$ the result follows.

Proof of Theorem 1

Set

$$S_n = \sum_{j=0}^{n} \left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) Z_{t-j}, \qquad R_n = \sum_{j=n+1}^{\infty} \left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) Z_{t-j},$$

$$Y_j = \left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) Z_{t-j},$$

Denote by G_j the distribution function of Y_j. First, it is easy to check that the moment generating function of Y_j exists and is given by

$$m_{G_j}(\gamma) = \sum_{i=0}^{j} m_F \left(\gamma \phi_1^i \phi_2^{j-i} \right) p_{j,i}.$$

Next note that $R_n = \phi_{(t)}\phi_{(t-1)} \dots \phi_{(t-n)}\alpha_{t-n-1}$. Using the independence between $\left\{ \phi_{(t)}, \phi_{(t-1)}, \dots, \phi_{(t-n)} \right\}$ and α_{t-n-1}, we can compute the moment generating function of R_n which we denote by m_{R_n},

$$m_{R_n}(\gamma) = \sum_{i=0}^{n+1} m_{\alpha_{t-n-1}} \left(\gamma \phi_1^i \phi_2^{n+1-i} \right) p_{n+1,i},$$

where m_{α_t} is the moment generating function of α_t. By stationarity of (α_t), we get

$$m_{R_n}(\gamma) = \sum_{i=0}^{n+1} m_{\alpha_t} \left(\gamma \phi_1^i \phi_2^{n+1-i} \right) p_{n+1,i}.$$

Using the rapidly variation of $\bar{F}$ and the tail balancing condition (6), we can establish that

$$\lim_{x\to\infty} \frac{\mathbb{P}\left(Y_j > x\right)}{\bar{F}(x)} = \lim_{x\to\infty} \sum_{k=0}^{j} \frac{\mathbb{P}(\phi_1^k \phi_2^{j-k} Z_{t-j} > x)}{\bar{F}(x)} p_{j,k} = \beta_j, \tag{22}$$

with

$$\beta_j = \begin{cases} q^j & \text{if} \quad \phi_1 = 1 \qquad \mid \phi_2 \mid < 1, \\ (p\delta_j + (1-p)\delta_{j+1})p^{-1}q^j & \text{if} \quad \phi_1 = -1 \qquad \mid \phi_2 \mid < 1, \\ (1-q)^j & \text{if} \quad \phi_2 = 1 \qquad \mid \phi_1 \mid < 1, \\ (p\delta_j + (1-p)\delta_{j+1})p^{-1}(1-q)^j & \text{if} \quad \phi_2 = -1 \qquad \mid \phi_1 \mid < 1, \\ 0 & \text{if} \quad |\phi_1| < 1, \qquad |\phi_2| < 1. \end{cases} \tag{23}$$

By Lemma 2, we have

$$\lim_{x\to\infty}\frac{\mathbb{P}(S_n>x)}{\mathbb{P}(Z_t>x)}=\sum_{i=1}^{n}\alpha_i m_{G_1*G_2*\ldots*G_{i-1}}(\gamma)m_{G_{i+1}}(\gamma)\ldots m_{G_n}(\gamma). \tag{24}$$

Moreover, by Lemma 3

$$\lim_{x\to\infty}\frac{\mathbb{P}(R_n>x)}{\overline{F}(x)}=0. \tag{25}$$

Combining (24) and (25) and applying again Lemma 2, we obtain the desired result.

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