# UNIFORM CONVERGENCE OF THE NON-WEIGHTED POVERTY MEASURES

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Abstract. We consider in this note the weak convergence, in the frame of the empirical processes theory, of the non-weighted poverty measures viewed as stochastic processes defined on some space of bounded functions and indexed by real numbers or monotone functions. The results include the asymptotic behavior of the Foster-Greer-Thorbecke process of poverty indices. We use them to follow up the poverty evolution in poor countries between two periods with appropriate curves.

## 1. Introduction

Poverty measurement and analysis are usually studied with the help of the Foster-Greer-Thorbecke (FGT) index (see [2]). It is computed for a population $\mathcal{P}$ of size $N$ and based on the income or expenditure variable $Y$. But a poverty line $Z$ should be set up before any measurement. Namely, authorities or economists must propose an income or expidenture threshold $Z$ under which the individuals of $\mathcal{P}$ are considered as poor ones. Let's consider the ordered incomes $Y_{j,N}$, $1\leq j \leq N$, of $\mathcal{P}$. The individual $j\in \mathcal{P}$, is a poor one if and only if

$$Y_{j,N} < Z$$

and then the exact number $Q$ of the poor individuals satisfies

$$Y_{Q,N} < Z \leq Y_{Q+1,N}.$$

From this notation, the FGT class of poverty measures is defined for $\alpha \geq 0$, by

$$J_N(\alpha) = \frac{1}{N}\sum_{j=1}^{Q}(\frac{Z-Y_{j,N}}{Z})^{\alpha}.$$

Besides this class of measures, many poverty indices are available in the literature since the pioneering work of Sen(1976) [6] who first derived poverty measures from an axiomatic point of view. A large survey of these indices is done in Zheng [8], along with their axiomatic properties. The poverty line determination, is also an important matter and several conflicted approaches are used by welfare's theory researchers (see [3] for a detailed discussion).

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In [1], [4], [5], the modern asymptotic theory of these indices when sampled from the population $\mathcal{P}$, has been raised. In [4], the general form of poverty measures

$$P_N = \frac{1}{a(Q)b(N)}\sum_{j=1}^{Q} c(N,Q,j)\ d\left(\frac{Z-Y_{j,N}}{Z}\right), \tag{1.1}$$

where a(·), b(·), c(·), d(·) are given measurable functions, has been introduced. This form includes the most famous poverty indices such as the Sen, Shorrocks and the Foster-Greer-Thorbecke ones.

To estimate $P_N$, a sample of size n, $Y_1, ..., Y_n$, is drawn from the population, and the following sample indice,

$$P_n = \frac{1}{a(q)b(n)}\sum_{j=1}^{q} c(n,q,j)\ d\left(\frac{Z-Y_{j,n}}{Z}\right),$$

is considered, where $q = q_n$ is the sample poor number and $Y_{1,n} \leq ... \leq Y_{n,n}$ are the order statistics based on $Y_1, ..., Y_n$. Under mild conditions on the distribution function $G$ of $Y$, the ultimate normality of $P_n$ is established and simulated. These results have been applied to compute poverty indices through confidence intervals. In particular, we have for the FGT indices with parameter $\alpha > 0$,

$$J_n(\alpha) = \frac{1}{n}\sum_{j=1}^{q}\left(\frac{Z-Y_{j,n}}{Z}\right)^{\alpha},$$

the following asymptotic normality

$$\sqrt{n}(J_n - D) \to N(0, \theta^2) \tag{1.2}$$

with

$$D = \int_0^{G(Z)} \left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds.$$

and

$$\theta^2 = Z^{-2}\int_0^{G(Z)}\int_0^{G(Z)} h(u)h(v)(u\wedge v - uv)\ du\ dv$$

where

$$h(s) = \alpha\ a(s)\ (\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z})^{\alpha-1}.$$

with this notation

$y_0 = \min\{x, G(x) > 0\}$, and $F(\cdot) = 1 - G(y_0 + 1/\cdot)$, $a(s) = (1/F^{-1}(1-s))'$,

.

We want to go further here and to complete this work by using a stochastic process approach. This is particularly interesting when we whish to compare the poverty situations of two areas or two prediods through a family of poverty measures. This kind of uniform comparison, which leads to curves comparison, is very more interesting than the comparison by one single poverty index. Thus we begin to investigate the behavior of the stochastic process $\{J_n(\alpha), \alpha \geq 0\}$ in $\ell^{\infty}(\mathbb{R}_+)$, the space of real bounded functions defined on $\mathbb{R}_+$. We aim to investigate the weak convergence of $J_n$ in the space $\ell^{\infty}(\mathbb{R}_+)$. The expected results yield many derived normality results because of the continuity theorem. Let's introduce further notation. We have for any $\alpha$,

$$J_n(\alpha) = \frac{1}{n}\sum_{j=1}^{n} f_\alpha(Y_j)$$

with

$$f_\alpha(x) = \left|\frac{Z-x}{Z}\right|^{\alpha} 1_{(x<Z)}.$$

Denote $\mathcal{F}$=$\{f_\alpha,\ \alpha \geq 0\}$ and consider for $f_\alpha \in \mathcal{F}$,

$$\mathbb{G}_n(\alpha) = \mathbb{G}_n(f_\alpha) = \sqrt{n}\{\frac{1}{n}\sum_{j=1}^{n} f_\alpha(Y_{j,n}) - \mathbb{E}f_\alpha(Y)\}.$$

Since $\mathcal{F}$ is bounded by $\mathrm{F}_0$=1, we have

$$\mathbb{E}f_\alpha(Y) = \int_0^Z (\frac{Z-x}{Z})^{\alpha}\ dG(x) \leq G(Z),$$

$$\mathbb{E}f_\alpha(Y)^2 = \int_0^Z (\frac{Z-x}{Z})^{2\alpha}\ dG(x) \leq G(Z)$$

and

$$\sup_{\alpha\geq 0} |f_\alpha(x) - \mathbb{E}f_\alpha(Y)| < \infty,$$

since for any $x \geq 0$, and $f \in \mathcal{F}$,

$$|f(x) - \mathbb{E}f(Y)| \leq |f(x)| + |\mathbb{E}f(Y)| \leq 1 + G(Z).$$

Finally $\{\mathbb{G}_n(\alpha),\ \alpha \geq 0\}$ lies in $\ell^{\infty}(\mathbb{R}_+)$. By Formula 2.1.2 in [7] p.81, we have the weak convergence in finite distribution, that is, for any finite set of elements $f_1, ..., f_k$ of $\mathcal{F}$,

$$(\mathbb{G}_n(f_1),\ \mathbb{G}_n(f_2),\ ...,\ \mathbb{G}_n(f_k)) \rightarrow_w N(0,\Sigma)$$

with $\sum_{ij} = \mathbb{P}(f_i(Y) - \mathbb{E}f_i(Y))(f_j(Y) - \mathbb{E}f_j(Y))$. Our results will also be given for the more general case of the nonweighted poverty measures in the form

$$J_n(\gamma) = \frac{1}{n}\sum_{j=1}^{q} \gamma(\frac{Z-Y_{j,n}}{Z}),$$

where $\gamma$ is an arbitrary element of some subclass $\mathcal{M}$ of the class of monotone functions $\gamma : [0,1] \mapsto [0,1]$. We will have to consider the stochastic process

$$\{\sqrt{n}(J_n(\gamma) - \mathbb{E}\gamma), \gamma \in \mathcal{M}\},$$

where

$$\mathbb{E}_\gamma = \int_0^Z \gamma(\frac{Z-x}{Z})\ dG(x)$$

We are now able to state our results and some of their applications.

## 2. RESULTS

**Theorem 1.** *For any $Z > 0$ such that $Z < sup\{x, G(x) < 1\}$ and for any distribution function $G$, $\{\mathbb{G}_n,\ \alpha \geq 0\}$ converges in distribution in $\ell^\infty(\mathbb{R}_+)$ to a Gaussian process with covariance function*

$$\Gamma(\alpha,\beta) = \mathbb{E}f_{\alpha+\beta}(Y) - \mathbb{E}f_\alpha(Y)\mathbb{E}f_\beta(Y) =: \mathbb{E}_{\alpha+\beta} - \mathbb{E}_\alpha\mathbb{E}_\beta,$$

*where $\mathbb{E}_\alpha = \mathbb{E}(f_\alpha(Y))$. Furthermore, the Glivenko-Cantelli theorem holds, that is*

$$\sup_{\alpha\geq 0} |J_n(\alpha) - \mathbb{E}_\alpha| \to 0,\ a.s,$$

*as $n\!\to 0$.*

This may be extended to a more general Theorem.

**Theorem 2.** *For any $Z > 0$ such that $Z > sup\{x, G(x) < 1\}$ and for any distribution function $G$, $\{\sqrt{n}(J_n(\gamma) - E\gamma), \gamma \in \mathcal{M}\}$ weakly converges in $\ell^\infty(\mathcal{F})$ to a Gaussian process with covariance function*

$$\Lambda(\gamma,\lambda) = \mathbb{P}(\gamma(\frac{Z-Y}{Z})_+) - \mathbb{E}_\gamma)(\lambda(\frac{Z-Y}{Z})_+) - \mathbb{E}_\lambda)),$$

*for $\mathcal{F} = \{t \hookrightarrow f(t) = \gamma(\frac{Z-t}{Z})1_{(t<Z)},\ \gamma \in \mathcal{M}\}$ whenever the map*

$$\omega \mapsto \sup_{\gamma\in\mathcal{M},\lambda\in\mathcal{M}} \left|\sum_{i=1}^n e_i \left[\gamma((\frac{Z-Y_i}{Z})_+) - \lambda(\frac{Z-Y_i}{Z})_+)\right]\right|(\omega) \tag{2.1}$$

*is measurable for any $n \geq 1$ and for any $(e_i)_{1\leq i\leq n} \subset \{-1,1\}^n$. Furthermore, $\mathcal{F}$ is a Glivenko-Cantelli class, that is*

$$(\sup_{\gamma\in\mathcal{M}} |J_n(\gamma) - \mathbb{E}_\gamma|)^* \to 0,\ a.s.$$

*This latter means that there exists a sequence of real random variables $v_n$ (that may be taken as $(\sup_{\gamma\in\mathcal{M}} |J_n(\gamma) - \mathbb{E}_\gamma|)^*$, the minimum measurable cover of $\sup_{\gamma\in\mathcal{M}} |J_n(\gamma) - \mathbb{E}_\gamma|$) such that*

- *For all $n \geq 1, 0 \leq \sup_{\gamma\in\mathcal{M}} |J_n(\gamma) - \mathbb{E}_\gamma| \leq v_n$*
- *$v_n \to 0$ a.s., as $n \to \infty$.*

**Remark 1.** *If the elements of $\mathcal{M}$ are continuous functions, then $\mathcal{F}$ is a subclass of the class of continuous functions C(0,1), which is separable. In this case (2.1) is measurable.*

**Remark 2.** *If the subclass $\mathcal{M}$ is pointwise measurable, theorem 2 holds since the measurability requirements are satisfied.*

**Applications**. The results allow the 95 percent uniform confidence intervals of the FGT poverty measures $J_N(\alpha)$ constructed from the asymptotic normality of $J_n(\alpha)$. Let us use the Senegalese data. In order to follow the poverty situation in the whole country from 1996 to 2001, we are able to compare the uniform confidence intervals when $\alpha$ takes 1000 values from 0 to 4. With the help of figure 1, we see that the poverty significantly decreased, pleading that the Senegalese are less poor. The poverty bridging is more striking for the Fatick area as shown in figure 2. The case is different for Kolda's area, the poorest district of Senegal, as illustrated in the figure 3. Although the poverty has clearly been alleviated, it has not decreased as much as in the whole Senegal.

FIGURE 1. Uniform estimation of FGT poverty indices from 1996 to 2001 in Senegal for $0 \leq \alpha \leq 4$

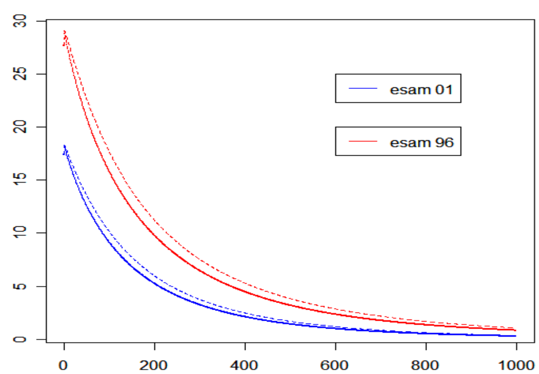


FIGURE 2. Uniform estimation of FGT poverty indices from 1996 to 2001 in Fatick for $0 \leq \alpha \leq 4$

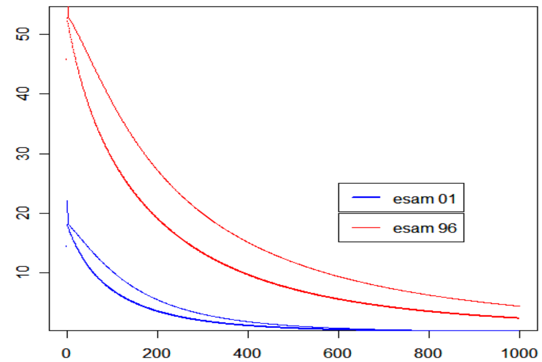
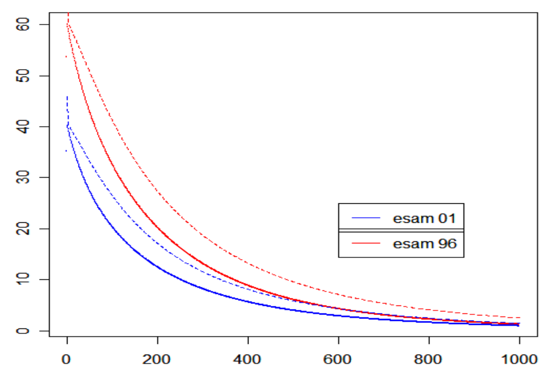



FIGURE 3. Uniform estimation of FGT poverty indices from 1996 to 2001 in Kolda for $0 \leq \alpha \leq 4$




## 3. PROOFS

We shall use the modern techniques of weak convergence in metric spaces, here in $\ell^\infty(\mathbb{R}_+)$, mainly available in [7]. By Theorem 2.5.2, we have to check three conditions in order to have the weak convergence. First $\mathcal{F}$ should satisfy the uniform entropy condition

$$\int_0^{+\infty} \sup_{\mathbb{Q}} \sqrt{\log N(\mathcal{F}, \varepsilon \left\| F_0 \right\|_{\mathbb{Q},2}, L_2(\mathbb{Q}))}\, d\varepsilon < \infty, \tag{C1}$$

where the supremum is taken over all the finitely discrete measures $\mathbb{Q}$ on $\mathbb{R}_+$ and $N(\mathcal{F}, \eta, L_2(\mathbb{Q}))$ is the $\eta-$covering number, that is the minimal number of balls of radius $\eta$ with respect to the $L_2(\mathbb{Q})$-norm, needed to cover $\mathcal{F}$. Next, for any $\delta > 0$, the families

$$\mathcal{F}_\delta = \{f - g,\ f, g \in \mathcal{F}, \|f - g\|_{P,2} \leq \delta\} \tag{C2}$$

and

$$\mathcal{F}_\infty^2 = \{f^2 - g^2,\ f, g \in \mathcal{F}\}. \tag{C3}$$

must be $\mathbb{P}_Y$-measurable. Finally we must check that

$$\mathbb{P}_Y^* F_0 < \infty, \tag{C4}$$

where $\mathbb{P}_Y^*$ stands for the outer-integral of $\mathbb{P}_Y$. But (C4) is immediate since $\mathrm{F}_0 = 1$ is measurable and then $\mathbb{P}_Y^* F_0 = \mathbb{P}_Y(1) = 1$. Recall that a family $\mathcal{F}$ is $\mathbb{P}_Y$-measurable if and only if for any $n \geq 1$ and for any sequence $(e_i)_{1 \leq i \leq n} \subset \{-1, 1\}^n$,

$$\omega \mapsto \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} e_i f(Y_i(\omega)) \right| \tag{3.1}$$

is measurable. Here, $\mathcal{F}_\delta$ and $\mathcal{F}_\infty^2$ are $\mathbb{P}_Y$-measurable whenever

$$\omega \mapsto \sup_{\alpha \geq 0,\ \beta \geq 0} \left| \sum_{i=1}^{n} e_i \left[ (\frac{Z - Y_i}{Z})_+^\alpha - (\frac{Z - Y_i}{Z})_+^\beta \right] \right| (\omega) \tag{3.2}$$

is measurable. But the supremun is achieved here through the rational values of $\alpha$ and $\beta$ so that the measurability of (3.2) is obtained. To assess (C1) we remark that $\mathcal{F}$ is included in the class $\mathcal{F}^0$ of monotone functions defined from $\mathbb{R}$ to $[0, 1]$. By Theorem 2.7.2 in [7], we have for any $\mathbb{Q}$, $\varepsilon > 0$, $r \geq 1$, for some constant $K > 0$ depending on r, that

$$\log N_{[]}(\mathcal{F}^0, \varepsilon, L_r(\mathbb{Q})) \leq K\ (1/\varepsilon), \tag{3.3}$$

where $N_{[]}(\mathcal{F}^0, \eta, L_2(\mathbb{Q}))$ is the $\eta-$bracketing number, that is the minimal number of $\eta-$brackets with respect to the $L_2(\mathbb{Q})$-norm, needed to cover $\mathcal{F}$. Recall that an $\eta-$bracket is a set of the form

$$[l, u] = \{f \in \mathcal{F}, l \leq f \leq u\} \ and \ \int (u - l)^2 d\mathbb{Q} \leq \eta^2\}.$$

Then for $0 \leq \varepsilon \leq 1$,

$$\sqrt{\log N(\mathcal{F}, \varepsilon, L_2(\mathbb{Q}))} \leq \sqrt{\log N(\mathcal{F}^0, \varepsilon, L_2(\mathbb{Q}))} \leq \sqrt{\log N_{[]}(\mathcal{F}^0, 2\varepsilon, L_2(\mathbb{Q}))} \leq \sqrt{K/2}\ \varepsilon^{-1/2}.$$

and, since $\mathrm{F}_0 = 1$, we also have for any $\varepsilon \geq 1$,

$$N(\mathcal{F}, \varepsilon, L_2(\mathbb{Q})) = 1.$$

Then

$$\int_0^{+\infty} \sup_{\mathbb{Q}} \sqrt{\log N(\mathcal{F}, \varepsilon \|F_0\|_{\mathbb{Q},2}, L_2(\mathbb{Q}))}\ d\varepsilon$$

$$= \int_0^1 \sup_{\mathbb{Q}} \sqrt{\log N(\mathcal{F}, \varepsilon \left\| F_0 \right\|_{\mathbb{Q},2}, L_2(\mathbb{Q}))}\, d\varepsilon \leq 2\sqrt{2K} < \infty.$$

All the needed conditions hold. Thus

$$\{\mathbb{G}_n(f), f \in \mathcal{F}\} \rightsquigarrow \mathbb{G} \text{ in } \ell^{\infty}(\mathcal{F}).$$

This proves the first assertion of the theorem, we use Theorem 2.4.1 in [7] and (3.3) to conclude that F is a Glivenko-Cantelli class, that is

$$\sup_{\alpha \geq 0} \left| \frac{1}{n} \sum_{j=1}^{n} f_{\alpha}(Y_j) - \mathbb{E}_{\alpha} \right|$$

which is measurable, converges to zero a.s, as $n \to \infty$.

We have finished to prove the first theorem. The proof of the second is a straightforward generalization of the first.

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