Journal of Statistical Planning and Inference 22 (1989) 127-136
North-Holland

A NOTE ON THE ASYMPTOTIC NORMALITY OF SUMS OF EXTREME VALUES

Gane Samb LO

Université Paris 6, L.S.T.A. T.45-55, E.3., 4 Place Jussieu, 75230 Paris 05, France

Received 15 January 1988; revised manuscript received 16 March 1988
Recommended by P. Deheuvels

Abstract: Let $X_1, X_2, \ldots,$ be a sequence of independent random variables with common distribution function F in the domain of attraction of a Gumbel extreme value distribution and for each integer $n \geq 1$, let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the first n of these random variables. Along with related results it is shown that for any sequence of positive integers $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, the sum of the upper k_n extreme values $X_{n-k_n+1,n} + \cdots + X_{n,n}$, when properly centered and normalized, converges in distribution to a standard normal random variable N(0, 1). These results constitute an extension of results by S. Csörgő and D.M. Mason (1985).

AMS Subject Classification: 62E20, 62G30, 60F05.

Key words: Order statistics; extreme values; Gumbel law; asymptotic normality.

1. Introduction

Let $X_1, X_2, \ldots,$ be a sequence of independent random variables with common distribution function F and for each integer $n \geq 1$, let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the first n of these random variables. Csörgő and Mason (1985, 1986) have recently shown among other results that if

$$1 - F(x) = L^*(x)x^{-a} \quad \text{as } x \to \infty, \tag{1}$$

where L^* is a slowly varying function at infinity and $a \geq 2$, or if F has exponential-like upper tails, meaning

$$\int_x^{\infty} (1 - F(y))\,\mathrm{d}y/(1 - F(x)) \to c \quad \text{as } x \to \infty, \tag{2}$$

where $0 < c < \infty$, then for any sequence of integers satisfying

$$1 \leq k_n \leq n, \quad k_n \to \infty \text{ and } k_n/n \to 0 \text{ as } n \to \infty, \tag{K}$$

there exist sequences $A_n > 0$ of normalizing constants and C_n of centering constants such that

$$A_n\left(\sum_{i=1}^{k_n} X_{n-i+1,n} - C_n\right) \xrightarrow{\mathrm{d}} \mathrm{N}(0,1), \quad \text{as } n \to \infty, \tag{3}$$

The case (1) is contained in the theorem of Csörgő and Mason (1986) and the case (2) is Theorem 1.5 of Csörgő and Mason (1985).

An application of Theorem 2.4.1 of de Haan (1970) (Lemma 1 below) combined with Fact 1.4 of Csörgő and Mason (1985) shows that (2) implies the existence of sequences of normalizing constants a_n and centering constants b_n such that

$$a_n^{-1}(X_{n,n}-b_n) \xrightarrow{\mathrm{d}} G \quad \text{as } n \to \infty, \tag{4}$$

where G is a Gumbel random variable with distribution function

$$P(G \le x) = \exp(-\mathrm{e}^{-x}) \quad \text{for } -\infty < x < \infty.$$

Whenever such sequences of constants can be chosen so that (4) holds, we say that F is in the domain of attraction of a Gumbel law, written $F \in D(\Lambda)$.

One of the purposes of this note is to show that (3) holds more generally than under condition (2), that is, $F \in D(\Lambda)$ is sufficient for (3) to hold. This will be a consequence of our main results stated in the next section. We shall also obtain some further extensions of the results of Csörgő and Mason. The proofs are given in Section 3.

2. Statement of main results

First we introduce some notations. Let

$$Q(s) = \inf\{x \colon F(x) \ge s\}, \quad \text{for } 0 < s \le 1,$$

with $Q(0) = Q(0+)$, denote the inverse or quantile function of F. Write

$$\sigma^2(s) = \int_{1-s}^{1} \int_{1-s}^{1} (\min(u,v) - uv)\, \mathrm{d}Q(u)\, \mathrm{d}Q(v), \quad \text{for } 0 \le s \le 1.$$

For any $0 < \beta < \infty$, set

$$c(s,\beta) = s^{-\beta} \int_{1-s}^{1} (1-u)^{\beta}\, \mathrm{d}Q(u), \quad \text{for } 0 \le s \le 1.$$

For convenience, when $\beta = 1$, we set $c(s) = c(s,1)$. (Refer to the next section for our integral convention.)

Let $D^*(\Lambda)$ denote the subclass of $D(\Lambda)$ consisting of all distribution functions F whose quantile function Q satisfies

$$Q(1-s) = a + \int_{s}^{1} u^{-1} r(u)\, \mathrm{d}u,$$

for all $s \ge 0$ sufficiently small, a is a fixed constant and r is a strictly positive function slowly varying at zero. The fact that $D^*(\Lambda)$ is a subclass of $D(\Lambda)$ follows from Theorem 2.4.1 of de Haan (1970).

For any sequence of positive integers k_n such that (K) holds and $F \in D(\Lambda)$, set for $n = 1, 2, \dots$,

$$\mu_n(k_n) = n \int_{1-k_n/n}^{1} Q(s)\,\mathrm{d}s.$$

The following theorem contains our main results.

Theorem. *On a rich enough probability space there exist a sequence of independent random variables* $X_1, X_2, \dots$, *with common distribution function* F *and a sequence of Brownian bridges* $B_1, B_2, \dots$, *such that for any sequence* k_n *satisfying* (K), *whenever* $F \in D(\Lambda)$,

$$\begin{aligned} &k_n^{-1/2} c(k_n/n)^{-1} \left\{ \sum_{i=1}^{k_n} X_{n-i+1,n} - \mu_n(k_n) \right\} \\ &\quad = -(n/k_n)^{1/2} c(k_n/n)^{-1} \int_{1-k_n/n}^{1} B_n(s)\,\mathrm{d}Q(s) + \mathrm{o}_p(1) := Z_n + \mathrm{o}_p(1), \end{aligned} \tag{5}$$

and whenever $F \in D^*(\Lambda)$,

$$\begin{aligned} &k_n^{1/2} c(k_n/n)^{-1} \{ X_{n-k_n,n} - Q(1-k_n/n) \} \\ &\quad = -(n/k_n)^{1/2} B_n(1-k_n/n) + \mathrm{o}_p(1) := Y_n + \mathrm{o}_p(1), \end{aligned} \tag{6}$$

and

$$\begin{aligned} &k_n^{-1/2} c(k_n/n)^{-1} \left\{ \sum_{i=1}^{k_n} X_{n-i+1,n} - k_n X_{n-k_n,n} - n \int_{1-k_n/n}^{1} r(1-s)\,\mathrm{d}s \right\} \\ &\quad = Z_n - Y_n + \mathrm{o}_p(1). \end{aligned} \tag{7}$$

Furthermore, the random variables on the left side of (5), (6) *and* (7), *respectively, converge in distribution to* N(0, 2), N(0, 1) *and* N(0, 1), *respectively, as* $n \uparrow \infty$.

Remark. With the choice $A_n = (2k_n)^{-1/2} c(k_n/n)^{-1}$ and $C_n = \mu_n(k_n)$, we see that our theorem implies (3) whenever $F \in D(\Lambda)$. Our theorem also extends Theorem 1.5, 1.7 and 2.1 and Corollary 2.5 of Csörgő and Mason (1985). The random variable on the left side of (7) is related to the Hill (1975) estimator of the tail index of a distribu- for this random variable was motivated by the work of Mason (1982) (see also Deheuvels, Haeusler and Mason (1988)).

3. Proof of the theorem

We use the following integral convention:
When $0 \le a \le b \le 1$, g is left continuous and f is right continuous,

$$\int_b^a f\,\mathrm{d}g = \int_{(a,b)} f\,\mathrm{d}g \quad \text{and} \quad \int_a^b g\,\mathrm{d}f = \int_{(a,b)} g\,\mathrm{d}f$$

whenever these integrals make sense as Lebesgue–Stieltjes integrals. In this case, the usual integration by parts formula

$$\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f = g(b)f(b) - g(a)f(a)$$

is valid.

The proof of our theorem will follow closely the proofs of the results of Csörgő and Mason (1985), substituting their technical lemmas concerning properties of the quantile functions of distribution functions satisfying (2), by those describing properties of the quantile functions of $F \in D(\Lambda)$. We therefore begin with these technical lemmas.

Lemma 1. *$F \in D(\Lambda)$ if and only if for each choice of $0 \le x, y, w, z < \infty$ fixed, $y \ne w$,*

$$\frac{Q(1-sx) - Q(1-sz)}{Q(1-sy) - Q(1-sw)} \to \frac{\log x - \log z}{\log y - \log w} \quad \textit{as } s \downarrow 0. \tag{8}$$

(This is Theorem 2.4.1 of de Haan (1970).)

Lemma 2. *Whenever $F \in D(\Lambda)$, $c(s, \beta)$ is slowly varying at zero for each choice of $0 < \beta < \infty$.*

Proof. We have to show that for each $0 < \lambda < \infty$ and $0 < \beta < \infty$,

$$c(\lambda s, \beta)/c(s, \beta) \to 1 \quad \text{as } s \downarrow 0. \tag{9}$$

Choose any $0 < \lambda < \infty$ and $0 < \theta < 1$. Then for all $s > 0$ small enough we have

$$\begin{aligned}\int_{1-\lambda s}^{1} (1-u)^\beta \,\mathrm{d}Q(u) &= \sum_{i=0}^{\infty} \int_{1-\lambda s\theta^i}^{1-\lambda s\theta^{i+1}} (1-u)^\beta \,\mathrm{d}Q(u) \\ &\le \sum_{i=0}^{\infty} \lambda^\beta s^\beta \theta^{i\beta} \{Q(1-\lambda s\theta^{i+1}) - Q(1-\lambda s\theta^i)\}.\end{aligned} \tag{10}$$

Applying Lemma 1 gives

$$\frac{Q(1-\lambda\theta u) - Q(1-\lambda u)}{Q(1-\theta u) - Q(1-u)} \to 1 \quad \text{as } u \downarrow 0. \tag{11}$$

Select any $0 < \varepsilon < \infty$. From (11) we have that, for all $s > 0$ sufficiently small, expression (10) is

$$\begin{aligned}&\le (1+\varepsilon) \sum_{i=0}^{\infty} \lambda^\beta s^\beta \theta^{i\beta} \{Q(1-s\theta^{i+1}) - Q(1-s\theta^i)\} \\ &\le \frac{(1+\varepsilon)\lambda^\beta}{\theta^\beta} \sum_{i=0}^{\infty} s^\beta \theta^{(i+1)\beta} \int_{1-s\theta^i}^{1-s\theta^{i+1}} \mathrm{d}Q(u) \\ &\le \frac{(1+\varepsilon)\lambda^\beta}{\theta^\beta} \int_{1-s}^{1} (1-u)\,\mathrm{d}Q(u).\end{aligned}$$

Thus for all $s>0$ sufficiently small,

$$c(\lambda s, \beta) \le \frac{(1+\varepsilon)}{\theta^{\beta}} c(s, \beta). \tag{12}$$

Observing that for all $s>0$ small enough,

$$\int_{1-\lambda s}^{1} (1-u)^{\beta} \, \mathrm{d}Q(u) \ge \sum_{i=0}^{\infty} \lambda^{\beta} s^{\beta} \theta^{(i+1)\beta} \{Q(1-\lambda s \theta^{i+1}) - Q(1-\lambda s \theta^{i})\},$$

we see that by an argument very much like the one just given, we have for all $s>0$ sufficiently small,

$$c(\lambda s, \beta) \ge (1-\varepsilon)\theta^{\beta} c(s, \beta). \tag{13}$$

Assertion (9) now follows from inequalities (12) and (13) by the fact that θ can be chosen arbitrarily close to one and ε arbitrarily close to zero. This completes the proof of Lemma 2.

The following lemma is related to Theorem 1.4.3.d of de Haan (1970) and its proof is based on a modification of the techniques used to prove this theorem. For details see Deheuvels et al. (1986).

Lemma 3. *Whenever $F \in D(\Lambda)$, there exists a constant $-\infty < b < \infty$ such that for all* $0 < s \le \frac{1}{2}$,

$$Q(1-s) = b - c(s) + \int_{s}^{1} u^{-1} c(u) \, \mathrm{d}u. \tag{14}$$

Lemma 4. *Whenever $F \in D(\Lambda)$, for each* $0 < x < \infty$,

$$\lim_{s \downarrow 0} \frac{Q(1-xs) - Q(1-s)}{c(s)} = -\log x. \tag{15}$$

Proof. Applying Lemma 3, we have for any $0 < x < \infty$ and for all s sufficiently small

$$\frac{Q(1-xs) - Q(1-s)}{c(s)} = \frac{c(s) - c(xs)}{c(s)} + \frac{1}{c(s)} \int_{xs}^{s} \frac{c(u)}{u} \, \mathrm{d}u.$$

Since c is slowly varying at zero, both

$$\inf\{c(u)/c(s) \colon u \in I(s)\} \to 1$$

and

$$\sup\{c(u)/c(s) \colon u \in I(s)\} \to 1$$

as $s \downarrow 0$, where $I(s)$ is the closed interval formed by xs and s. From these two facts the proof of Lemma 4 follows immediately.

Lemma 5. *Whenever $F \in D(\Lambda)$, for any* $0 < \beta < \infty$,

$$c(s, \beta)/c(s) \to 1/\beta \quad \text{as } s \downarrow 0. \tag{16}$$

Proof. Let $\hat{Q}(1-s)=Q(1-s^{1/\beta})$. Since by Lemma 1 for any choice of $0<x, y<\infty$, $y\neq 1$,

$$\frac{\hat{Q}(1-xu)-\hat{Q}(1-u)}{\hat{Q}(1-yu)-\hat{Q}(1-u)}=\frac{Q(1-x^{1/\beta}u^{1/\beta})-Q(1-u^{1/\beta})}{Q(1-y^{1/\beta}u^{1/\beta})-Q(1-u^{1/\beta})}$$

converges as $u\downarrow 0$ to $\log x/\log y$, we conclude that $\hat{Q}\in D(\Lambda)$.

Let

$$\hat{c}(s)=s^{-1}\int_{1-s}^{1}(1-u)\,\mathrm{d}\hat{Q}(u), \quad \text{for } 0<s<1.$$

A change of variables shows that $\hat{c}(s^{\beta})=c(s,\beta)$ for $0<s<1$. Thus

$$\begin{aligned}\frac{c(s,\beta)}{c(s)}&=\frac{\hat{c}(s^{\beta})}{c(s)}\\&=\frac{Q(1-2s)-Q(1-s)}{c(s)}\times\frac{\hat{Q}(1-(2s)^{\beta})-\hat{Q}(1-s^{\beta})}{Q(1-2s)-Q(1-s)}\\&\quad\times\frac{\hat{c}(s^{\beta})}{\hat{Q}(1-2^{\beta}s^{\beta})-\hat{Q}(1-s^{\beta})}\end{aligned}$$

which by Lemmas 1 and 4 converges to $1/\beta$ as $s\downarrow 0$, completing the proof of Lemma 5.

Lemma 6. *Whenever* $F\in D(\Lambda)$,

$$\sigma^2(s)/(2sc^2(s))\to 1 \quad \textit{as } s\downarrow 0. \tag{17}$$

Proof. The proof is based on Lemma 5 and follows almost exactly as the proof of Lemma 3.3 of Csörgő and Mason (1985). Therefore, the details are omitted.

The proof of the following lemma is an easy consequence of the Karamata representation for a slowly varying function.

Lemma 7. *Let* a_n *be any sequence of positive constants such that* $a_n\to 0$ *and* $na_n\to\infty$. *Also let L be any slowly varying function at zero. Then for any* $0<\beta<\infty$,

$$n^{-\beta}L(1/n)/((a_n)^{\beta}L(a_n))\to 0 \quad \textit{as } n\uparrow\infty. \tag{18}$$

We now describe the probability space on which the assertions of the theorem are assumed to hold. M. Csörgő, S. Csörgő, Horváth and Mason (1986) have constructed a probability space $(\Omega, \mathcal{F}, P)$ carrying a sequence $U_1, U_2, \ldots,$ of independent random variables uniformly distributed on $(0,1)$ and a sequence $B_1, B_2, \ldots,$ of Brownian bridges such that for the empirical process

$$\alpha_n(s)=n^{1/2}\{G_n(s)-s\}, \quad 0\le s\le 1,$$

and the quantile process

$$\beta_n(s)=n^{1/2}\{s-U_n(s)\},\quad 0\le s\le 1,$$

where

$$G_n(s)=n^{-1}\#\{k\colon 1\le k\le n,\ U_k\le s\},$$

and, with $U_{1,n}\le\cdots\le U_{n,n}$ denoting the order statistics corresponding to $U_1,\ldots,U_n$.

$$U_n(s)=\begin{cases}U_{k,n} & \text{if } (k-1)/n<s\le k/n,\ k=1,\ldots,n,\\ U_{1,n} & \text{if } s=0,\end{cases}$$

we have

$$\sup_{0\le s\le 1} n^{\nu_1}|\alpha_n(s)-\bar{B}_n(s)|/(1-s)^{-\nu_1+1/2}=\mathrm{O}_p(1). \tag{19}$$

with $\bar{B}_n(s)=B_n(s)$ for $1/n\le s\le 1-1/n$ and zero elsewhere and

$$\sup_{0\le s\le 1-1/n} n^{\nu_2}|\beta_n(s)-B_n(s)|/(1-s)^{-\nu_2+1/2}=\mathrm{O}_p(1), \tag{20}$$

where ν_1 and ν_2 are any fixed number such that $0\le\nu_1<\frac{1}{4}$ and $0\le\nu_2\le\frac{1}{2}$. The statement in (19) follows from Theorem 2.1, while the statement in (20) is easily inferred from Corollaries 2.1 and 4.2.2 of the above paper.

Thoughout the remainder of the proof of our theorem, we assume that we are on the probability space of Csörgő et al. (1986). Since the sequence of random variables $X_1, X_2, \ldots,$ is equal in distribution to $Q(U_1), Q(U_2), \ldots,$ we can and do assume that the first sequence is equal to the second.

First assume $F\in D(\Lambda)$. We shall establish (5). Applying integration by parts we see that the left side of (5) equals

$$-(n/k_n)^{1/2}c(k_n/n)^{-1}\int_{1-k_n/n}^{1}\alpha_n(s)\,\mathrm{d}Q(s)+nk_n^{-1/2}\int_{U_{n-k_n,n}}^{1-k_n/n}(1-G_n(s)-k_n/n)\frac{\mathrm{d}Q(s)}{c(k_n/n)}$$

$$:=\Delta_{1,n}+\Delta_{2,n}.$$

We shall first show that

$$\Delta_{1,n}=Z_n+R_n$$

with $R_n=\mathrm{o}_p(1)$.

From (19) we have for any $0<\nu<\frac{1}{4}$,

$$\sup_{0\le s\le 1}|\alpha_n(s)-\bar{B}_n(s)|/(1-s)^{-\nu+1/2}=\mathrm{O}_p(n^{-\nu}). \tag{21}$$

Notice that for any such ν,

$$|R_n|\le\sup_{0\le s\le 1}\frac{|\alpha_n(s)-\bar{B}_n(s)|}{(1-s)^{-\nu+1/2}}\left(\frac{n}{k_n}\right)^{1/2}\int_{1-k_n/n}^{1}(1-s)^{-\nu+1/2}\,\mathrm{d}Q(s)/c(k_n/n)$$

$$+\left|\left(\frac{n}{k_n}\right)^{1/2}\int_{1-1/n}^{1}B_n(s)\,\mathrm{d}Q(s)/c(k_n/n)\right|:=R_{1,n}+R_{2,n}.$$

From (21), we obtain

$$R_{1,n} = \mathrm{O}_p(n^{-v}) n^{v} \frac{c(k_n/n, \frac{1}{2} - v)}{c(k_n/n)} k_n^{-v},$$

which by Lemma 5 equals $\mathrm{o}_p(1)$.

Also

$$ER_{2,n}^2 = \sigma^2(1/n) \bigg/ \left(\frac{k_n}{n} c^2\left(\frac{k_n}{n}\right)\right),$$

which by Lemma 6 is

$$\sim 2\sigma^2(1/n)/\sigma^2(k_n/n) \quad \text{as } n \to \infty.$$

From Lemmas 2 and 6 we infer that $\sigma^2(s)$ is regularly varying of exponent one at zero. Hence, by Lemma 7,

$$\sigma^2(1/n)/\sigma^2(k_n/n) \to 0 \quad \text{as } n \to \infty,$$

which yields $R_{2,n} = \mathrm{o}_p(1)$.

Thus we have proved $R_n = \mathrm{o}_p(1)$.

Next we show that $\Delta_{2,n} = \mathrm{o}_p(1)$. Choose any $1 < \lambda < \infty$ and set

$$T_n(\lambda) = n k_n^{-1/2} c(k_n/n)^{-1} |1 - G_n(1 - k_n/n) - k_n/n| \{Q(r_n^+(\lambda)) - Q(r_n^-(\lambda))\},$$

where

$$r_n^-(\lambda) = 1 - \frac{\lambda k_n}{n} \quad \text{and} \quad r_n^+(\lambda) = 1 - \frac{k_n}{\lambda n}.$$

Notice that since for all s in the closed interval formed by $U_{n-k_n,n}$ and $1 - k_n/n$,

$$|1 - G_n(s) - k_n/n| \le |1 - G_n(1 - k_n/n) - k_n/n|,$$

we have for any $1 < \lambda < \infty$,

$$\liminf_{n\to\infty} P(|\Delta_{2,n}| \le |T_n(\lambda)|) \ge \liminf_{n\to\infty} P(r_n^-(\lambda) \le U_{n-k_n,n} \le r_n^+(\lambda)\}.$$

Since (K) implies (cf. Balkema and de Haan (1975)) that

$$n(1 - U_{n-k_n,n})/k_n \xrightarrow{P} 1 \quad \text{as } n \to \infty, \tag{22}$$

the lower bound in the above inequality equals one.

Hence for each $1 < \lambda < \infty$,

$$\lim_{n\to\infty} P(|\Delta_{2,n}| \le |T_n(\lambda)|) = 1. \tag{23}$$

Observe for each $1 < \lambda < \infty$,

$$ET_n(\lambda) \le \left\{ Q\left(1 - \frac{k_n}{\lambda_n}\right) - Q\left(1 - \frac{\lambda k_n}{n}\right)\right\} / c(k_n/n).$$

Applying Lemma 4 we see that this last expression converges to $2\log\lambda$, which yields

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty} ET_n(\lambda)=0. \tag{24}$$

The fact that $\Delta_{2,n}=\mathrm{o}_p(1)$ now follows by an elementary argument based on (23) and (24). This completes the proof of (5).

Next consider (6). Notice that since $F\in D^*(\Lambda)$,

$$c(s)=s^{-1}\int_{1-s}^{1} r(1-u)\,\mathrm{d}u.$$

Thus since $r>0$ and slowly varying at zero Theorem 1.2.1 of de Haan (1970) gives

$$r(s)/c(s)\to 1 \quad \text{as } s\downarrow 0. \tag{25}$$

The left side of (6) equals

$$-\frac{k_n^{1/2}}{c(k_n/n)}\int_{k_n/n}^{1-U_{n-k_n,n}}\frac{r(u)}{u}\,\mathrm{d}u=-k_n^{1/2}\frac{r(k_n/n)}{c(k_n/n)}\{\log(1-U_{n-k_n/n})-\log(k_n/n)\},$$
$$-\frac{k_n^{1/2}}{c(k_n/n)}\int_{k_n/n}^{1-U_{n-k_n,n}}(r(u)-r(k_n/n))\frac{\mathrm{d}u}{u}:=\Delta_{1,n}^*+\Delta_{2,n}^*.$$

The same argument based on (20) as given in Csörgő and Mason (1985) shows that

$$-k_n^{1/2}\{\log(1-U_{n-k_n,n})-\log(k_n/n)\}=Y_n+\mathrm{o}_p(1).$$

Therefore by (25) and the fact that $Y_n=\mathrm{o}_p(1)$ we have

$$\Delta_{1,n}^*=Y_n+\mathrm{o}_p(1).$$

Since r is slowly varying at zero we get for each $1<\lambda<\infty$ as $n\to\infty$,

$$\sup\left\{|r(s)-r(k_n/n)|/c(k_n/n)\colon \frac{k_n}{\lambda_n}\le s\le\frac{\lambda k_n}{n}\right\}\to 0. \tag{26}$$

The fact that $\Delta_{2,n}^*=\mathrm{o}_p(1)$ now follows easily from $Y_n=\mathrm{O}_p(1)$, (22) and (26) completing the proof of (6).

Since $F\in D^*(\Lambda)$ we have

$$\mu_n(k_n)-k_nQ(1-k_n/n)=n\int_{1-k_n/n}^{1} r(1-s)\,\mathrm{d}s.$$

Assertion (7) is now a direct consequence of (5) and (6).

Finally we prove the convergence in distribution of Z_n, Y_n and Z_n-Y_n to $\mathrm{N}(0,2)$, $\mathrm{N}(0,1)$ and $\mathrm{N}(0,1)$, respectively, as $n\to\infty$. Notice that the Z_n random variable in (5) is normal with mean zero and second moment $\sigma^2(k_n/n)/(k_nc^2(k_n)/n)$, which by Lemma 6 converges to 1 as $n\to\infty$. The Y_n random variable in (6) is normal with mean zero and second moment $1-k_n/n\to 1$ as $n\to\infty$.

The Z_n-Y_n random variable in (7) is normal with mean zero. Applying Lemmas 5 and 6 it is easy to verify that $E(Z_n-Y_n)^2\to 1$ and $n\to\infty$.

This completes the proof of the theorem.

Acknowledgement

The author is extremely grateful to David Mason, Erich Haeusler and Paul Deheuvels, for helping him to complete this work.

References

Balkema, A. and L. de Haan (1975). Limit laws for order statistics. In: P. Révész, Ed., *Limit Theorems of Probability Theory*, Coll. Math. Soc. Bolyai, Vol. 11, 17-22.

Csörgő, M., S. Csörgő, L. Horváth and D.M. Mason (1986). Weighted empirical and quantile processes, Ann. Prob. **11**, 31-85.

Csörgő, S. and D.M. Mason (1985). Central limit theorems for sums of extreme values, *Math. Proc. Cambridge Philos. Soc.* **98**, 547-558.

Csörgő, S. and D.M. Mason (1986). The asymptotic distribution of sums of extreme values from a regularly varying distribution, *Ann. Probab.* **14**, 974-983.

Deheuvels, P., E. Haeusler and D.M. Mason (1988). Almost sure convergence of the Hill estimator. *Math. Proc. Cambridge Philos. Soc.* **104**, 371-381.

Deheuvels, P., E. Haeusler and D.M. Mason (1989). Laws of the iterated logrithm for sums of extreme values in the domain of attraction of a Gumbel law. *Bull. Sci. Math.*, to appear.

de Haan, L. (1970). *On Regular Variation and its Application to the Weak Convergence of Sample Extremes.* Mathematical Centre Tract No. 32 (Amsterdam).

Hill, B.M. (1975). A simple general approach to inference about the tail of a distribution. *Ann. Statist.* **3**, 1163-1174.

Lo, G.S. (1986). Asymptotic behavior of Hill's estimate and applications. *J. Appl. Probab.* **23**, 922-936.

Mason, D.M. (1982). Laws of large numbers for sums of extreme values. *Ann. Probab.* **10**, 756-764.