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Ratio of Generalized Hill's Estimator and Its Asymptotic Normality Theory

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Abstract—We present a statistical process depending on a continuous time parameter τ whose each margin provides a Generalized Hill's estimator. In this paper, the asymptotic normality of the finite-dimensional distributions of this family are completely characterized for $\tau > 1/2$ when the underlying distribution function lies on the maximum domain of attraction. The ratio of two different margins of the statistical process characterizes entirely the whole domain of attraction. Its asymptotic normality is also studied. The results permit in general to build a new family of estimators for the extreme value index whose asymptotic properties can be easily derived. For example, we give a new estimate of the Weibull extreme value index and we study its consistency and its asymptotic normality.

Key words: order statistics, empirical and quantile processes, Brownian bridges, maximum domain of attraction, asymptotic normality, extreme value theory.

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1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ denote a sample from a distribution function F. The estimation of the law of the maximum $X_{n,n} = \max(X_1, \ldots, X_n)$ is one of the most important questions of the extreme value theory. Recall that $X_{n,n}$ converges in type to some random variable Z with nondegenerate distribution function $H(\cdot)$ if and only if there exist two sequences of real numbers $(a_n > 0,\ n \geq 1)$ and $(b_n,\ n \geq 1)$ such that

$$\lim_{n\to\infty} \mathbb{P}(X_{n,n} \leq a_n x + b_n) = H(x), \quad \forall x \in \mathbb{R}.$$

It is then said that F is in the domain of attraction of H, denoted by $F \in D(H)$. It is well known that H belongs to the type of one of the following three dfs:

$$\begin{aligned} &\text{Gumbel}: && \Lambda(x) = \exp(-e^{-x}), && x \in \mathbb{R}, \\ &\text{Fréchet}: && \varphi_\alpha(x) = \exp(-x^{-\alpha}), && x > 0, \quad \alpha > 0, \\ &\text{Weibull}: && \psi_\gamma(x) = \exp(-(-x)^{\gamma}), && x \leq 0, \quad \gamma > 0. \end{aligned}$$

The interested reader is referred to [6], [7], [2], [15], [11], [1], [10] or [20] for a general introduction to extreme value theory and applications. The characterization of these three domains received very much consideration in the two past decades. At the same time, many empirical discrimination results within the extremal domain are available. We mean by discrimination results those that give three different behaviors for the three domains, so that we may decide from them what extremal domain applies. First, Mason [19] gave a complete characterization of the Fréchet type with Hill's statistic [14]. Tiago de Oliveira [23], Smith [22], Dekkers *et al.* [8], Hasofer and Wang [13] also proposed statistical tests for the selection between these three types. In the same spirit, Lô [18] introduced a class of nine statistics which

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characterizes completely all the extremal domain. Next, Diop and Lô [9] introduced the following family, depending on the continuous parameter τ, $0 < \tau < \infty$, whose each margin can arise as a Generalized Hill's estimator (when l is fixed):

$$T_n(\tau) = k_n^{-\tau} \sum_{j=l_n+1}^{k_n} j^{\tau}(\log X_{n-j+1,n} - \log X_{n-j,n}),$$

where l_n and k_n are sequences of integers such that

$$0 < l_n < k_n < n, \qquad l_n/k_n \to 0 \qquad \text{and} \qquad k_n/n \to 0 \qquad \text{as} \quad n \to \infty, \tag{K}$$

and where $X_{1,n} \leq \ldots \leq X_{n,n}$ are the order statistics based on $X_1, \ldots, X_n$. They established the strong consistency of this estimator when the underlying distribution function belongs to $D(\varphi_\alpha)$, $\alpha > 0$.

We aim to characterize here the asymptotic normality of this family of statistics. In this paper, we shall focus on the asymptotic normality of their finite-dimensional distributions. The asymptotic normality of the ratio of any two margins is also studied.

Before we go any further, let us give some notation. We assume $F(1) = 0$. The sequence $Y_1,\ Y_2,\ \ldots$ is related to the X_i's by $Y_i = \log X_i$, $G(y) = F(e^y)$. In the tail estimation problem, it is advantageous for a number of reasons to transform the original data by taking logarithms. Negative observations, not relevant to estimating the right tail anyway, are neglected.

The function $G^{-1}(\cdot) = \inf\{x,\ G(x) \geq \cdot\}$ is the left-continuous generalized inverse of G. We will have the following representations. First, we have for $F \in D(\varphi_\alpha)$,

$$G^{-1}(1-u) = \log c(1+f(u)) - \frac{\log u}{\alpha} + \int_u^1 b(t)t^{-1}\,dt, \qquad 0 < u < 1, \tag{1}$$

next for $F \in D(\psi_\gamma)$, $y_0 = \sup\{x, G(x) < 1\} < +\infty$

$$y_0 - G^{-1}(1-u) = c(1+f(u))u^{1/\gamma} \exp\left(\int_u^1 b(t)t^{-1}\,dt\right), \qquad 0 < u < 1, \tag{2}$$

and finally, for $F \in D(\Lambda)$,

$$G^{-1}(1-u) = d - s(u) + \int_u^1 s(t)t^{-1}dt,\ 0 < u < 1, \tag{3}$$

where $s(u) = c(1+f(u))\exp\left(\int_u^1 b(t)t^{-1}\,dt\right)$, $0 < u < 1$, is a slowly varying function in a neighborhood of 0 (SVZ). In each of these formulae, $(f(u), b(u)) \to (0,0)$ as $u \to 0$. The representations (1) and (2) are the Karamata's representation, while (3) is that of de Haan.

Throughout the paper, $U_1,\ U_2,\ \ldots, U_n$ will be a sequence of independent and uniform random variables on $(0,\ 1)$ and $(U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n})$ will denote the corresponding order statistics. Let $\mathbb{U}_n(\cdot)$ be the empirical distribution function based on $U_1,\ U_2,\ \ldots, U_n$ $(n = 1, 2, \ldots)$. We then put for $0 < \tau < \infty$, $n \geq 1$,

$$\widetilde{x}_n = G^{-1}(1 - U_{k_n+1,n}), \qquad \widetilde{z}_n = G^{-1}(1 - U_{l_n+1,n}),$$
$$x_n = G^{-1}(1 - k_n/n), \qquad z_n = G^{-1}(1 - l_n/n),$$
$$\widetilde{\mu}_n(\tau) = (n/k_n)^{\tau} \int_{\widetilde{x}_n}^{\widetilde{z}_n} (1 - G(t))^{\tau}\,dt, \qquad \mu_n(\tau) = (n/k_n)^{\tau} \int_{x_n}^{z_n} (1 - G(t))^{\tau}\,dt,$$
$$r(\tau, x, z) = (1 - G(x))^{-\tau} \int_x^z (1 - G(t))^{\tau}\,dt,$$

$$r(\tau, x) \equiv r(\tau, x, y_0), \qquad r(x) \equiv r(1, x, y_0)$$

with $x \le z \le y_0$, where $y_0 = \sup\{x,\ G(x) < 1\}$ denotes the right endpoint of G.

We may without loss of generality and do assume the general representations for the empirical distribution function (df) G_n based on $Y_1,\ Y_2,\ \ldots, Y_n$ and for the statistics $Y_{1,\,n} \le \ldots \le Y_{n,\,n}$ by their uniform counterparts:

$$\{1 - G_n(x),\ x \in \mathbb{R},\ n \ge 1\} = \{U_n(1 - G(x)),\ x \in \mathbb{R},\ n \ge 1\},$$
$$\{Y_{n-i+1,\,n},\ 1 \le i \le n,\ n \ge 1\} = \{G^{-1}(1 - U_{i,\,n}),\ 1 \le i \le n,\ n \ge 1\}.$$

In what follows, the main results are stated in Section 2. Section 3 contains an application of the main results. In Section 4, we prove the different results. Section 4 concludes the paper.

2. MAIN RESULTS

In the sequel, we also need the following functions for $\tau > 0$, $\rho > 0$:

$$\sigma_1^2(\tau) = \frac{2\tau^2(\gamma+1)^2}{(\gamma\tau+1)(\gamma(2\tau-1)+2)}, \tag{4}$$

$$\sigma_2^2(\tau) = \frac{(\gamma+1)^2(\tau\gamma^2-\gamma+2)}{\gamma^2(\gamma\tau+1)(\gamma(2\tau-1)+2)}, \tag{5}$$

$$\sigma_3^2(\tau,\rho) = 2\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2\left\{\frac{\tau^2(\gamma\tau+1)}{\gamma(2\tau-1)+2} + \frac{\rho^2(\gamma\rho+1)}{\gamma(2\rho-1)+2} - \tau\rho\frac{\gamma(\rho+\tau)+2}{\gamma(\rho+\tau-1)+2}\right\}, \tag{6}$$

$$\sigma_4^2(\tau,\rho) = \gamma^{-2}\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2\left\{\frac{(\tau\gamma^2-\gamma+2)(\gamma\tau+1)}{\gamma(2\tau-1)+2} + \frac{(\rho\gamma^2-\gamma+2)(\gamma\rho+1)}{\gamma(2\rho-1)+2} - 2\frac{\gamma^3\tau\rho+\gamma(\tau+\rho-1)+2}{\gamma(\rho+\tau-1)+2}\right\}, \tag{7}$$

$$\Gamma_1(\tau,\rho) = \frac{\tau\rho(\gamma+1)^2(\gamma(\tau+\rho)+2)}{(\gamma(\tau+\rho-1)+2)(\gamma\tau+1)(\gamma\rho+1)},$$

$$\Gamma_2(\tau,\rho) = \frac{(\gamma+1)^2(\gamma^3\tau\rho+\gamma(\tau+\rho-1)+2)}{\gamma^2(\gamma(\tau+\rho-1)+2)(\gamma\tau+1)(\gamma\rho+1)}.$$

We also put $\mathfrak{D} = D(\Lambda) \cup D(\varphi_\alpha) \cup D(\psi_\gamma)$. Any formula (variance, covariance,...) with the parameter $\gamma > 0$ is true when $F \in D(\psi_\gamma)$ and the corresponding result for the domain $D(\varphi_\alpha) \cup D(\Lambda)$ is obtained by letting $\gamma \to +\infty$, unless the contrary is specified.

We are now able to give our results for these statistical processes for $0 < T < \infty$,

$$\{\mathbb{S}_n(\tau),\ n > 1\} = \left\{k_n^{1/2}(T_n(\tau) - \widetilde{\mu}_n(\tau))/r(x_n),\ 0 < \tau < T,\ n > 1\right\}$$

and

$$\{\mathbb{W}_n(\tau),\ n > 1\} = \left\{k_n^{1/2}(T_n(\tau) - \mu_n(\tau))/r(x_n),\ 0 < \tau < T,\ n > 1\right\}.$$

Theorem 1. *Assume that* (K) *holds and let $F \in \mathfrak{D}$, $\tau > 1/2$, $\rho > 1/2$ and $\tau \neq \rho$. Then*

$$(\mathbb{S}_n(\tau),\ \mathbb{S}_n(\rho)) \ \to_d\ \mathfrak{N}_2\left(0, \begin{bmatrix} \sigma_1^2(\tau) & \Gamma_1(\tau,\rho) \\ \Gamma_1(\tau,\rho) & \sigma_1^2(\rho) \end{bmatrix}\right) \tag{8}$$

and

$$k_n^{1/2}\left(\frac{T_n(\tau)}{T_n(\rho)} - \frac{\widetilde{\mu}_n(\tau)}{\widetilde{\mu}_n(\rho)}\right) \ \to_d\ \mathfrak{N}(0, \sigma_3^2(\tau,\rho)). \tag{9}$$

The centering coefficients are random in this theorem. In order to be able to replace them by nonrandom ones, we need the following regularity condition:

$$\gamma_n(k_n) = k_n^{1/2}\{f(U_{k_n+1,n}) - f(k_n/n)\} \rightarrow_p 0 \qquad \text{as} \quad n \rightarrow \infty. \tag{RC}$$

Precisely, we have:

Theorem 2. *Assume that* (K) *holds and let* $F \in \mathfrak{D}$, $\tau > 1/2$, $\rho > 1/2$, *and* $\tau \neq \rho$. *Assume also that* $(k_n,\, l_n)$ *satisfies*:

H_1 : *There exists* η, $0 \leq \eta < 1/2$, $l_n^{\tau} k_n^{-\tau+\frac{1}{2}+\eta} \rightarrow 0$ *as* $n \rightarrow +\infty$.

Then,

(i) $W_n(\tau) \ \rightarrow_d \ \mathfrak{N}(0, \sigma_2^2(\tau))$ *if and only if* (RC) *holds.*

(ii) *Moreover, whenever* (RC) *holds,*

$$\big(\mathbb{W}_n(\tau),\, \mathbb{W}_n(\rho)\big) \ \ \rightarrow_d \ \ \mathfrak{N}_2\left(0, \begin{bmatrix} \sigma_2^2(\tau) & \Gamma_2(\tau,\rho) \\ \Gamma_2(\tau,\rho) & \sigma_2^2(\rho) \end{bmatrix}\right) \tag{10}$$

and

$$k_n^{1/2}\left(\frac{T_n(\tau)}{T_n(\rho)} - \frac{\mu_n(\tau)}{\mu_n(\rho)}\right) \ \ \rightarrow_d \ \ \mathfrak{N}(0, \sigma_4^2(\tau,\rho)). \tag{11}$$

Remark 1. According to Theorem 1 and Lemma 1 below, the couple $(T_n(\tau), T_n(\rho)/T_n(\tau))$ tends in probability to $(0, \tau/\rho)$, $(1/(\tau\alpha), \tau/\rho)$ or $(0, (\gamma\tau+1)/(\gamma\rho+1))$ according to the three cases $F \in D(\Lambda)$, $F \in D(\varphi_\alpha)$ or $F \in D(\psi_\gamma)$. These results clearly give a total separation of the whole domain of attraction. The theorems then propose statistical tests based of the consistency results.

Remark 2. It is important to see that the variances given in (4)–(7) are defined only for $\tau > 1/2$ and $\rho > 1/2$.

We are now going further to see the conditions that will enable us to get rid of the coefficients in Theorem 2 and replace them by more explicit ones leading to more sound statistical applications. In order to give a refinement of the theorems by replacing the centering constants in (10) by zero, $1/(\tau\alpha)$ or $(y_0 - G^{-1}(1 - k_n/n))/(\gamma\tau + 1)$ according to $F \in D(\Lambda)$, $F \in D(\varphi_\alpha)$ or $F \in D(\psi_\gamma)$ and in (11) by $(\gamma\rho + 1)/(\gamma\tau + 1)$, we need some additional conditions. The following theorem gives the result when $F \in D(\psi_\gamma)$.

Theorem 3. *Assume that* (K), (RC) *and* H_1 *hold and let* $F \in D(\psi_\gamma)$, $\tau > 1/2$, $\rho > 1/2$, *and* $\tau \neq \rho$. *Assume also that the following hypotheses hold*:

H_2 : $k_n^{1/2} \sup\limits_{x_n < z < y_0} |f(z)| \rightarrow 0$ *as* $n \rightarrow \infty$;

H_3 : $k_n^{1/2}\left[\left(\dfrac{n}{k_n}(1 - G(x_n)\right)^{\tau} - 1\right] \rightarrow 0$ *as* $n \rightarrow \infty$.

Then

$$\frac{k_n^{1/2}}{r(x_n)}\left[T_n(\tau) - \frac{y_0 - x_n}{\gamma\tau + 1}\right] \ \rightarrow \ \mathfrak{N}(0, \sigma_2^2(\tau)) \tag{12}$$

and

$$k_n^{1/2}\left[\frac{T_n(\tau)}{T_n(\rho)} - \frac{\gamma\rho + 1}{\gamma\tau + 1}\right] \ \rightarrow_d \ \mathfrak{N}(0, \sigma_4^2(\tau,\rho)). \tag{13}$$

Remark 3.

(1) If we suppose that the distribution function G is continuous and strictly increasing in a neighborhood of the right endpoint y_0, then H_3 holds.

(2) If we attempt to have a nonrandom centering sequence in Theorems 2 and 3, only f makes problems. In earlier studies of $T_n(\tau)$ when $\tau = 1$ for instance, Hall [12], Csörgő and Mason [5], Lô [17] imposed $f \equiv 0$. In this case the hypothesis H_2 and the assumption (RC) hold.

3. APPLICATION

As an application of the obtained asymptotic results, we present a new family of estimates of the Weibull extreme value index depending on two continuous parameters τ and ρ. These estimates are based on the ratio of Generalized Hill's estimates. Therefore, their weak consistency and asymptotic normality follow directly from Theorem 3. Their asymptotic as well as finite-sample performance will be compared to classical ones in future work. Indeed, according to Theorem 3, if we set $R_n(\tau,\rho)=\frac{T_n(\tau)}{T_n(\rho)}$, we have

$$R_n(\tau,\rho)\to_p \frac{\gamma\rho+1}{\gamma\tau+1}. \tag{14}$$

Using (14), we can define a new estimate of the Weibull extreme value index. The estimate $\widehat{\gamma}_n$ is defined as the solution of the following equation

$$R_n(\tau,\rho)=\frac{\widehat{\gamma}_n\rho+1}{\widehat{\gamma}_n\tau+1},$$

$$\widehat{\gamma}_n=\frac{R_n(\tau,\rho)-1}{\rho-\tau R_n(\tau,\rho)}, \tag{15}$$

so that $\widehat{\gamma}_n$ depends on τ and ρ. For every τ and ρ, $\widehat{\gamma}_n$ arises as an estimator of the parameter γ when $F\in D(\psi_\gamma)$. The consistency and asymptotic normality of $\widehat{\gamma}_n$ follow directly from (14) and Theorem 3 respectively. Precisely, we have

$$k^{1/2}\left[\widehat{\gamma}_n-\gamma\right]\ \to_d\ \mathfrak{N}(0,\sigma_5^2(\tau,\rho)),$$

where

$$\sigma_5^2(\tau,\rho)=\frac{(\gamma\tau+1)^4}{(\rho-\tau)^2}\sigma_4^2(\tau,\rho).$$

4. PROOFS OF THE MAIN RESULTS

We need two lemmas. First, define for $-\infty<x<z<y_0$, $\tau>0$, $\rho>0$

$$w(\tau,\rho,x,z)=\frac{1}{(1-G(x))^{\tau+\rho}}\int\limits_x^z\int\limits_y^z(1-G(t))^\tau(1-G(y))^\rho\,dtdy,$$

with $w(\tau,\rho,x,y_0)\equiv w(\tau,\rho,x)$.

Lemma 1. *Let $\tau>0$, $\rho>0$, $\gamma>0$.*

(i) *If $F\in D(\Lambda)\cup D(\varphi_\alpha)$, then $1-(1-G)^\tau\in D(\Lambda)$ and $r(\tau,G^{-1}(1-u))$ is slowly varying at zero and*

$$w(\tau,\rho,x,z)\sim w(\tau,\rho,x)\sim r(\tau,x)r(\tau+\rho,x) \tag{16}$$

as $x\to y_0$, $z\to y_0$ and $\frac{1-G(z)}{1-G(x)}\to 0$.

(ii) *If $F\in D(\Psi_\gamma)$, then $1-(1-G)^\tau\in D(\Psi_{\gamma\tau})$ and $r(\tau,G^{-1}(1-u))$ is regularly varying at zero with exponent γ^{-1}. Moreover,*

$$w(\tau,\rho,x,z)\sim w(\tau,\rho,x)\sim e(\gamma,\tau,\rho)r(\tau,x)r(\tau+\rho,x) \tag{17}$$

as $x\to y_0$, $z\to y_0$, $\frac{1-G(z)}{1-G(x)}\to 0$, where

$$e(\gamma,\tau,\rho)=\{\gamma(\tau+\rho)+1\}/\{\gamma(\tau+\rho)+2\}.$$

(iii) *If $F\in\mathfrak{D}$, then $\big(1-G(G^{-1}(1-u))\big)/u\to 1$ as $u\to 0$.*

(iv) *If $F\in\mathfrak{D}$, then $r(\tau,x)/r(\rho,x)\to(\gamma\rho+1)/(\gamma\tau+1)$ as $x\to y_0$, for $\tau>0$, $\rho>0$.*

Proof. (i) Let $F \in D(\Lambda) \cup D(\varphi_\alpha)$. It is well known that $G \in D(\Lambda)$. Then using Karamata's representation we prove that $1-(1-G)^\tau \in D(\Lambda)$. Next Lemma 1 in Lô [16] yields for $\lambda > 0$, $\lambda \neq 1$,

$$\frac{G^{-1}(1-\lambda u) - G^{-1}(1-u)}{r(G^{-1}(1-u))} \to -\log\lambda \quad \text{as} \quad u \to 0$$

and Lemma 4 in Lô [17] implies

$$\frac{G^{-1}(1-\lambda u) - G^{-1}(1-u)}{s(u)} \to -\log\lambda \quad \text{as} \quad u \to 0,$$

where s is SVZ defined in (3), so that $r(G^{-1}(1-u))$ is SVZ. Using Theorem 2.8.1 in De Haan [6], we can write $\frac{r(\tau, G^{-1}(1-u))}{r(G^{-1}(1-u))} \to \frac{1}{\tau}$ as $u \to 0$. Since $r(G^{-1}(1-u))$ is SVZ for $G \in D(\Lambda)$, it follows easily that $r(\tau, G^{-1}(1-u))$ is SVZ. It remains to prove (16). Set for $\rho > 0$ and $\tau > 0$,

$$1 - G^*(x) = (1-G(x))^\rho \int_x^{y_0} (1-G(t))^\tau \, dt.$$

Further, by Theorem 2.8.1 in De Haan [6], $t(x) = t$, $r(\tau+\rho, x)/r(x) \to 1/(\tau+\rho)$ as $x \to y_0$. Next, since $G \in D(\Lambda)$, by Hospital's rule we have

$$\Bigg(\int_{x+t(x)r(x)}^{y_0} (1-G(t))^\tau \, dt \Bigg) \Big/ \Bigg(\int_x^{y_0} (1-G(t))^\tau \, dt \Bigg) \to e^{-\frac{t\tau}{\rho+\tau}}$$

as $x \to y_0$. Therefore

$$\begin{aligned}
&\frac{1 - G^*\left(x + t(x) r(\tau+\rho, x)\right)}{1 - G^*(x)} \\
&= \left(\frac{1 - G(x + t(x)r(x))}{1 - G(x)} \right)^\rho \Bigg(\int_{x+t(x)r(x)}^{y_0} (1-G(t))^\tau \, dt \Bigg) \Big/ \Bigg(\int_x^{y_0} (1-G(t))^\tau \, dt \Bigg) \\
&\to e^{-\frac{t\rho}{\rho+\tau}} e^{-\frac{t\tau}{\rho+\tau}} = e^{-t}.
\end{aligned}$$

Finally, by using again Lemma 2.5.1 in [6], we have $G^* \in D(\Lambda)$ and

$$r(\tau+\rho, x) \sim \frac{1}{1-G^*(x)} \int_x^{y_0} (1 - G^*(t)) \, dt = w(\tau, \rho, x)/r(\tau, x) \tag{18}$$

as $x \to y_0$. To complete the proof of (16), it remains to show that

$$w(\tau, \rho, x, z) \sim w(\tau, \rho, x) \tag{19}$$

as $z \wedge x \uparrow y_0$ whenever $(1-G(z))/(1-G(x)) \to 0$ as $z \wedge x \uparrow y_0$. Indeed, we have

$$\begin{aligned}
w(\tau, \rho, x, z) &= \frac{1}{(1-G(x))^{\tau+\rho}} \int_x^z \int_y^z (1-G(t))^\tau (1-G(y))^\rho \, dt dy \\
&= w(\tau, \rho, x)(1 - E_2 - E_3)
\end{aligned}$$

with

$$\begin{aligned}
E_2 &= \Bigg(\int_x^z \int_z^{y_0} (1-G(t))^\tau (1-G(y))^\rho \, dt dy \Bigg) \Big/ \Bigg(\int_x^{y_0} \int_y^{y_0} (1-G(t))^\tau (1-G(y))^\rho \, dt dy \Bigg) \\
&= \frac{(1-G(x))^\tau (1-G(z))^\rho r(\tau, x) r(\rho, z)}{(1-G(x))^{\tau+\rho} w(\tau, \rho, x)}
\end{aligned}$$

and

$$E_3 = \Big(\int_z^{y_0} \int_y^{y_0} (1 - G(t))^{\tau} (1 - G(y))^{\rho} \, dtdy \Big) \Big/ \Big(\int_x^{y_0} \int_y^{y_0} (1 - G(t))^{\tau} (1 - G(y))^{\rho} \, dtdy \Big).$$

By (18),

$$E_2 \sim \Big(\frac{1 - G(z)}{1 - G(x)}\Big)^{\rho} \frac{r(\rho, z)}{r(\tau + \rho, x)} \sim \frac{\tau + \rho}{\rho} \Big(\frac{1 - G(z)}{1 - G(x)}\Big)^{\rho} \frac{r(z)}{r(x)}. \tag{20}$$

But

$$\frac{r(z)}{r(x)} \leq \frac{r(z)}{z - x} \to 0 \tag{21}$$

whenever $(1 - G(z))/(1 - G(x)) \to 0$ as $z \wedge x \uparrow y_0$. Combining (20) and (21) yields $E_2 \to 0$. By the same arguments,

$$\begin{aligned} E_3 &\leq \Big(\int_z^{y_0} \int_y^{y_0} (1 - G(t))^{\tau} (1 - G(y))^{\rho} \, dtdy \Big) \Big/ \Big(\int_x^{z} \int_y^{z} (1 - G(t))^{\tau} (1 - G(y))^{\rho} \, dtdy \Big) \\ &\leq 2 \frac{w(\tau, \rho, z)}{(z - x)^2} \sim 2 \frac{r(\tau + \rho, z)}{z - x} \frac{r(\tau, z)}{z - x} \sim \frac{2}{\tau(\tau + \rho)} \Big(\frac{r(z)}{x - z} \Big)^2 \to 0. \end{aligned}$$

This completes the proof of Point (i) of the Lemma.

(ii) Let $F \in D(\Psi_\gamma)$. Then $G \in D(\Psi_\gamma)$. Use the Karamata representation of G:

$$1 - G(t) = c(t)(y_0 - t)^{\gamma} \exp \int_1^{\frac{1}{y_0 - t}} \frac{b(t)}{t} \, dt, \tag{22}$$

with $b(t) \to 0$ and $c(t) \to c > 0$. We get for all $\tau > 0$,

$$(1 - G(t))^{\tau} = c(t)^{\tau} (y_0 - t)^{\gamma\tau} \exp \int_1^{\frac{1}{y_0 - t}} \frac{\tau b(t)}{t} \, dt.$$

This means that $1 - (1 - G)^{\tau} \in D(\Psi_{\tau\gamma})$. Now, one easily gets for any $\tau > 0$ and $\rho > 0$, as $x \to y_0$,

$$r(\tau, x) = \frac{(y_0 - x)}{(\gamma\tau + 1)} (1 + o(1)) \tag{23}$$

and

$$w(\tau, \rho, x) = \frac{(y_0 - x)^2}{(\gamma\tau + 1)(\gamma(\tau + \rho) + 2)} (1 + o(1)).$$

But

$$r(\tau + \rho, x) r(\tau, x) = \frac{(y_0 - x)^2}{(\gamma\tau + 1)(\gamma(\tau + \rho) + 1)} (1 + o(1)).$$

Comparing these two last formulas gives (17) for $z = y_0$. Recall that (19) has been proved for any G when $F \in \mathfrak{D}$. Then (17) is completely proved. To finish, recall that one has, by virtue of (23),

$$r(\tau, G^{-1}(1 - u)) \sim \frac{y_0 - G^{-1}(1 - u)}{(\gamma\tau + 1)},$$

which is regularly varying at zero with exponent $1/\gamma$ by the classical Karamata's representation of $G \in D(\Psi_\gamma)$ (derived from (22)). The proof of Point (ii) of the lemma is finished.

(iii) It is easily checked that either $G(G^{-1}(1-u)) = 1-u$ or $1-u$ lies on some constancy interval of G^{-1}, say $]G(x-),\ G(x)]$, so that $G^{-1}(1-u) = x$ and hence $1 \geq (1 - G(G^{-1}(1-u)))/u \geq (1 - G(x))/(1-G(x-))$. Either $G \in D(\Psi_\gamma)$ and consequently Karamata's representation holds:

$$1 - G(x) = c(x)(y_0 - x)^\gamma \exp\Bigg(\int\limits_1^{1/(y_0-x)} p(t)t^{-1}\,dt \Bigg), \qquad x < y_0,$$

where $c(x) \to c$, $c > 0$, as $x \to y_0$ and $p(t) \to 0$ as $t \to +\infty$, or $G \in D(\Lambda)$ and by de Haan–Balkema's representation (see Smith [22]),

$$1 - G(x) = c(x) \exp\Bigg(\int\limits_{-\infty}^{x} l(t)^{-1}\,dt \Bigg), \qquad -\infty < x < y_0,$$

where $c(x) \to 1$ as $x \to y_0$ and l admits a derivative $l'(x)$ such that $l'(x) \to 0$ as $x \to y_0$. In both cases,

$$(1 - G(x))/(1 - G(x-)) = c(x)/c(x-) \to 1 \qquad \text{as} \quad x \to y_0.$$

This completes the proof.

(iv) The proof directly follows from Theorem 2.8.1 of De Haan [6]. □

Lemma 2. *Assume that* (K) *holds. Then there exists a probability space carrying a sequence of independent random variables uniformly distributed on* $(0,1)$ *denoted* $U_1, U_2, \ldots$ *and a sequence of Brownian bridges* $\{B_n(s),\ 0 \leq s \leq 1\}$, $n = 1, 2, \ldots$, *such that for all* ν, $0 \leq \nu < 1/4$, *and for all* $\tau > 0$,

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{n^{1/2}\{(U_n(s))^\tau - s^\tau\} - \tau s^{\tau-1} B_n(s)}{s^{\tau-1+1/2-\nu}} = O_p(n^{-\nu}),$$

where $U_n(\cdot)$ *is the empirical distribution function based on* $U_1, U_2, \ldots, U_n$, $n \geq 1$.

Proof. Csörgő *et al.* [3] have constructed a probability space on which the following approximation holds for $0 \leq \nu < 1/4$ (see [21], p. 501):

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{n^{1/2}[U_n(s) - s] - B_n(s)}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}). \tag{24}$$

Now, by using twice the mean-value theorem, we get

$$\begin{aligned} &n^{1/2}\{(U_n(s))^\tau - s^\tau\} - \tau s^{\tau-1} B_n(s) \\ &\quad = \tau s^{\tau-1}(\alpha_n(s) - B_n(s)) + \tau(\tau-1)\alpha_n(s)(\zeta_n'(s) - s)(\zeta_n''(s))^{\tau-2} \\ &\quad \equiv R_n(1,s) + R_n(2,s), \end{aligned}$$

where $\alpha_n(\cdot)$ defined by $n^{1/2}(U_n(s) - s)$ is the empirical uniform process and

$$\{\zeta_n'(s), \zeta_n''(s)\} \subset [\min(U_n(s), s), \max(U_n(s), s)].$$

It is immediate in view of (24) that,

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{|R_n(1,s)|}{s^{\tau-1+1/2-\nu}} = O_p(\tau n^{-\nu}).$$

Furthermore, it is clear that

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} |s^{-1/2} B_n(s)| = O_p(1).$$

Remark also that (24) is also true for $\nu = 0$. Thus

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} |s^{-1/2}\alpha_n(s)| \leq O_p(1) + \sup_{U_{1,n} \leq s \leq U_{n,n}} |s^{-1/2} B_n(s)| = O_p(1).$$

Hence,

$$\sup_{U_{1,n}\le s\le U_{n,n}} \frac{|R_n(2,s)|}{s^{\tau-1}} = \tau(\tau-1) \sup_{U_{1,n}\le s\le U_{n,n}} n^{-1/2}\left|\frac{\alpha_n(s)^2}{s}\right| \sup_{U_{1,n}\le s\le U_{n,n}} |\zeta_n''(s)/s|^{\tau-2}$$
$$= O_p(\tau(\tau-1)n^{-1/2}).$$

Finally,

$$\sup_{U_{1,n}\le s\le U_{n,n}} \frac{|R_n(2,s)|}{s^{\tau-1+1/2-\nu}} = O_p(n^{-\nu}).$$

This completes the proof of the lemma. □

For convenience, we use in the sequel the following notation, for $\tau > 0$ and $0 < s \le 1$:

$$\widetilde{\alpha}_n(\tau,s) = n^{1/2}\{(U_n(s))^\tau - s^\tau\}$$

and

$$\widetilde{B}_n(\tau,s) = \tau s^{\tau-1} B_n(s).$$

We are now able to prove our results.

Proof of Theorem 1. It is easily seen that

$$T_n(\tau) = \left(\frac{n}{k_n}\right)^\tau \int_{\widetilde{x}_n}^{\widetilde{z}_n} (1-G_n(s))^\tau\, ds$$

and then

$$\frac{k_n^{1/2}}{r(1,x_n)}\{T_n(\tau) - \widetilde{\mu}_n(\tau)\} = \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{\widetilde{z}_n} n^{1/2}\big((1-G_n(s))^\tau - (1-G(s))^\tau\big)\, ds. \tag{25}$$

Thus, the right-hand side of (25) can be decomposed into four parts:

$$\frac{k_n^{1/2}}{r(1,x_n)}\{T_n(\tau) - \widetilde{\mu}_n(\tau)\} = N_n(1,\tau) + R_n(1) + R_n(2) + R_n(3), \tag{26}$$

where the terms $R_n(j)$, $j = 1,2,3$, and $N_n(1,\tau)$ are specified and handled in the coming steps.

Step 1: We first show that

$$R_n(1) = \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{x_n} \widetilde{B}_n(\tau, 1-G(t))\, dt \ \to_p\ 0.$$

Proof. For any integer d fixed with $1 < d < n$, $U_{d,n}$ has the same law as S_d/S_{n+1}, where S_n is the partial sum of the first n elements of a sequence of independent random variables with standard exponential distribution (see [21], Proposition 1, p. 335). Hence, by the law of large numbers, $nU_{k_n+1,n}/k_n \to_p 1$ as $k_n \to \infty$, while for l fixed, $nU_{l+1,n}/l = O_p(1)$ as $n \to \infty$. This implies that for any $\epsilon > 0$,

$$\forall\lambda > 1 \qquad (\exists n_1 > 0) \quad \forall n \ge n_1, \quad k_n/n\lambda \le U_{k_n+1,n} \le \lambda k_n/n$$

holds with probability at least greater than $1-\epsilon$ (denoted w.p.g. $1-\epsilon$) and

$$\forall\lambda > 1 \qquad (\exists n_2 > 0) \qquad \forall n \ge n_2, \qquad l_n/n\lambda \le U_{l_n+1,n} \le \lambda l_n/n \tag{27}$$

w.p.g. $1-\epsilon$. We also use the notation $t_n(\cdot) = G^{-1}(1-\cdot/n)$. Hence, for $t > (\widetilde{x}_n \wedge k_n/n)$,

$$1 - G(t) \le U_{k_n+1,n} \wedge k_n/n \le k_n/(\lambda n),$$

w.p.g. $1-\epsilon$, for large $n \geq n_1$. Thus, w.p.g. $1-\epsilon$, for $n \geq n_1$

$$\mathbb{E}|R_n(1)| \leq 2\tau \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{t_n(k_n\lambda)}^{t_n(k_n)} (1-G(t))^{\tau-1}(G(t)(1-G(t)))^{1/2}\,dt$$
$$\leq \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \Big\{(1-G(t_n(k_n\lambda)))^{\tau-1/2} r(\tau-1/2, t_n(k_n\lambda))$$
$$- (1-G(t_n(k_n)))^{\tau-1/2} r(\tau-1/2, t_n(k_n))\Big\},$$

which tends to

$$\frac{\lambda^{\tau-1/2}}{\gamma(\tau-1/2)+1} - \frac{1}{\gamma(\tau-1/2)+1},$$

by virtue of Points (i) and (ii) of Lemma 1 and this for any $\lambda > 1$. Then, as $\lambda \to 1$, this term is zero. Therefore a probabilistic argument can show that $\mathbb{E}|R_n(1)| \to 0$ and then $R_n(1) \to_p 0$ by Markov's inequality.

Step 2: Next we show that

$$R_n(3) = \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{\widetilde{z}_n} \big(\widetilde{\alpha}_n(\tau, 1-G(t)) - \widetilde{B}_n(\tau, 1-G(t))\big)\,dt \;\to_p\; 0.$$

Proof. For $\widetilde{x}_n < t < \widetilde{z}_n$, we have $U_{1,n} < 1-G(t) \leq U_{n,n}$, so that by Lemma 2

$$R_n(3) = O_p(n^{-\nu}) \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{\widetilde{z}_n} (1-G(t))^{\tau-\frac{1}{2}-\nu}\,dt.$$

For $\tau > 1/2$, we choose $\nu > 0$ such that $\tau - \frac{1}{2} - \nu > 0$. By Lemma 1 (iii) and (iv), it follows that

$$|R_n(3)| \leq O_p(n^{-\nu}) \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{y_0} (1-G(t))^{\tau-\frac{1}{2}-\nu}\,dt$$
$$= O_p(n^{-\nu})(1-G(\widetilde{x}_n))^{-\nu} \frac{r(\tau-\frac{1}{2}-\nu, \widetilde{x}_n)}{r(1,x_n)} = O_p(k_n^{-\nu}).$$

Step 3: Now we show that

$$R_n(2) = \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{z_n}^{\widetilde{z}_n} \widetilde{B}_n(\tau, 1-G(t))\,dt \;\to_p\; 0.$$

Proof. By using (27) and the same methods as before, we get, w.p.g. $1-\epsilon$, for $n \geq n_2$ and some $\lambda > 0$,

$$\mathbb{E}|R_n(2)| \leq 2\tau \frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)} \int_{t_n(l/\lambda)}^{t_n(l)} (1-G(t))^{\tau-1}(G(t)(1-G(t)))^{1/2}\,dt$$
$$= o(1)(l/k_n)^{\tau-1/2} \frac{r(1,z_n)}{r(1,x_n)} \frac{(n/l_n)^{\tau-1/2}}{r(1,z_n)} \Big\{(1-G(t_n(l_n/\lambda)))^{\tau-1/2} r(\tau-1/2, t_n(l_n/\lambda))$$
$$- (1-G(t_n(l_n)))^{\tau-1/2} r(\tau-1/2, t_n(l_n))\Big\}.$$

But the same methods used just above show that

$$\frac{(n/l_n)^{\tau-1/2}}{r(1,z_n)}\Big\{(1-G(t_n(l_n/\lambda)))^{\tau-1/2}r(\tau-1/2,t_n(l_n/\lambda))-(1-G(t_n(l_n)))^{\tau-1/2}r(\tau-1/2,t_n(l_n))\Big\}$$

converges to

$$\frac{1}{(\gamma(\tau-1/2)+1)\lambda^{\tau-1/2+1/\gamma}}-\frac{1}{(\gamma(\tau-1/2)+1)}$$

and

$$\Big(\frac{l_n}{k_n}\Big)^{\tau-1/2}\frac{r(1,z_n)}{r(1,x_n)}\to 0$$

in view of Points (i) and (ii) of Lemma 1. Hence $\mathbb{E}|R_n(2)|\to 0$ and thus $R_n(2)\to_p 0$.

Step 4: Il remains to show that

$$N_n(1,\tau)=\frac{(n/k_n)^{\tau-1/2}}{r(1,x_n)}\int_{x_n}^{z_n}\widetilde{B}_n(\tau,1-G(t))\,dt\ \sim\ \mathfrak{N}(0,\sigma_1^2(\tau)).$$

It is clear that $N_n(1,\tau)$ is a centered normal random variable with variance

$$\sigma_{1,n}^2(\tau)=\frac{(n/k_n)^{2\tau-1}}{r(1,x_n)^2}\tau^2\int_{x_n}^{z_n}\int_{x_n}^{z_n}(1-G(t))^{\tau-1}(1-G(s))^{\tau-1}$$
$$\times\Big\{\min(1-G(t),1-G(s))-(1-G(t))(1-G(s))\Big\}\,dsdt.$$

We now split this variance into

$$\sigma_{1,n}^2(\tau)=\sigma_{1,n,1}^2(\tau)+\sigma_{1,n,2}^2(\tau), \tag{28}$$

where

$$\sigma_{1,n,1}^2(\tau)=\frac{(n/k_n)^{2\tau-1}}{r(1,x_n)^2}\tau^2\int_{x_n}^{z_n}\int_{x_n}^{s}G(t)(1-G(t))^{\tau-1}(1-G(s))^{\tau}\,dsdt$$

and

$$\sigma_{1,n,2}^2(\tau)=\frac{(n/k_n)^{2\tau-1}}{r(1,x_n)^2}\tau^2\int_{x_n}^{z_n}\int_{s}^{z_n}G(s)(1-G(t))^{\tau}(1-G(s))^{\tau-1}\,dsdt.$$

By using Points (i), (ii), and (iii) of Lemma 1, one easily gets

$$\sigma_{2,n,1}^2(\tau)\sim\frac{\tau^2(\gamma+1)^2}{(\gamma(2\tau-1)+2)(\gamma\tau+1)}.$$

Next, an integration by parts in $\sigma_{1,n,1}^2(\tau)$ with

$$u=\int_{x_n}^{s}(1-G(t))^{\tau-1}G(t)\,dt\qquad\text{and}\qquad dv=d\Big(\int_{s}^{z_n}(1-G(t))^{\tau}\,dt\Big)$$

yields $\sigma_{1,n,1}^2(\tau)=\sigma_{2,n,1}^2(\tau)$. And this leads to $\sigma_{1,n}^2(\tau)\to\sigma_1^2(\tau)$ as $n\to\infty$. This establishes the asymptotic normality of the margins of $\mathbb{S}_n(\tau)$. Moreover, the finite-dimensional distributions are obtained from those of the Brownian bridge. It remains to compute the covariance function, that is the covariance between $\mathbb{S}_n(\tau)$ and $\mathbb{S}_n(\rho)$, given by the limit of

$$\mathbb{E}(N_n(1,\tau)N_n(1,\rho)) = \frac{\tau\rho(n/k_n)^{\tau+\rho-1}}{r(1,x_n)^2}$$
$$\times \int_{x_n}^{z_n}\int_{x_n}^{z_n} (1-G(s))^{\tau-1}(1-G(s))^{\rho-1}\Big[(1-G(t))\Lambda(1-G(s)) - (1-G(t))(1-G(s)\Big]\,dsdt.$$

The same techniques as used in (28), cutting and integrating by parts, yield

$$E\big(N_n(1,\tau)N_n(1,\rho)\big) = \frac{\tau\rho e_3(\gamma) r(\tau,x_n) r(\tau+\rho-1,x_n)}{r(1,x_n)^2}(1+o(1)) + \frac{\tau\rho e_3(\gamma) r(\rho,x_n) r(\tau+\rho-1,x_n)}{r(1,x_n)^2}(1+o(1)) \tag{29}$$

with $e_3(\gamma) = \frac{\gamma(\tau+\rho-1)+1}{\gamma(\tau+\rho-1)+2}$. The limit of the left-hand side of (29) is $\Gamma_1(\tau,\rho)$ by virtue of Lemma 1. This achieves the proof of (8). It remains to prove (9). By (8) and Lemma 1, we have the following one-term expansion for any $\tau > 1/2$,

$$T_n(\tau) = \widetilde{\mu}_n(\tau) + \frac{N(1,\tau)r(1,x_n)}{k_n^{1/2}} + o_p(r(1,x_n)k_n^{-1/2})$$
$$= \widetilde{\mu}_n(\tau)\left(1 + \frac{N(1,\tau)(\gamma\tau+1)}{(\gamma+1)k_n^{1/2}}\right) + o_p(k_n^{-1/2}).$$

Then

$$T_n(\rho)^{-1} = \widetilde{\mu}_n(\rho)^{-1}\left(1 - \frac{N(1,\rho)(\gamma\rho+1)}{(\gamma+1)k_n^{1/2}}\right) + o_p(k_n^{-1/2})$$

and, since $\widetilde{\mu}_n(\tau)\widetilde{\mu}_n(\rho)^{-1} \sim (\gamma\rho+1)/(\gamma\tau+1)$ by Lemma 1, we have

$$\frac{T_n(\tau)}{T_n(\rho)} = \frac{\widetilde{\mu}_n(\tau)}{\widetilde{\mu}_n(\rho)} + \frac{(\gamma\rho+1)N(1,\tau)}{(\gamma+1)k_n^{1/2}} - \frac{(\gamma\rho+1)^2}{(\gamma+1)(\gamma\tau+1)k_n^{1/2}} + o_p(k_n^{-1/2}).$$

This implies

$$\sqrt{k_n}\left(\frac{T_n(\tau)}{T_n(\rho)} - \frac{\widetilde{\mu}_n(\tau)}{\widetilde{\mu}_n(\rho)}\right) = \frac{(\gamma\rho+1)N(1,\tau)}{(\gamma+1)} - \frac{(\gamma\rho+1)^2}{(\gamma+1)(\gamma\tau+1)}N(1,\rho) + o_p(1)$$
$$=: N(4,\tau,\rho) + o_p(1). \tag{30}$$

We finish by evaluating the variance of $N(4,\tau,\rho)$. But $\mathbb{E}(N(1,\tau)^2) \sim \sigma_1^2(\tau)$ and $\mathbb{E}(N(1,\tau)N(1,\rho)) \sim \Gamma_1(\tau,\rho)$. Hence it is easy to show that the asymptotic form of $\mathbb{E}(N(4,\tau,\rho)^2)$ is $\sigma_3^2(\tau,\rho)$. The proof of Theorem 1 is now complete. □

Proof of Theorem 2. From (26) and the different steps, we get for $\tau > 1/2$,

$$\mathbb{W}(\tau) = N_n(1,\tau) + D_n + R_n(4) + o_p(1) \tag{31}$$

with

$$R_n(4) = \frac{\sqrt{k}}{r(1,x_n)}(n/k_n)^\tau \int_{z_n}^{\widetilde{z}_n} (1-G(t))^\tau\,dt$$

and

$$D_n = \frac{\sqrt{k_n}}{r(1,x_n)}(n/k_n)^\tau \int_{\widetilde{x}_n}^{x_n} (1-G(t))^\tau\,dt.$$

By (27), for any $\epsilon > 0$, for some $\lambda > 1$ and for $n \geq n_2$,

$$|R_n(4)| \leq \frac{n^{\tau}}{k_n^{\tau-1/2} r(1, x_n)} \int\limits_{t_n(l_n)}^{t_n(l_n/\lambda)} (1 - G(t))^{\tau}\, dt$$

$$= \frac{n^{\tau}}{k_n^{\tau-1/2} r(1, x_n)} \Big\{(1 - G(t_n(l_n)))^{\tau} r(\tau, t_n(l_n)) - (1 - G(t_n(l_n/\lambda)))^{\tau} r(\tau, t_n(l_n/\lambda))\Big\} \quad (32)$$

w.p.g. $1 - \epsilon$. The latter term goes to zero whenever

$$l_n^{\tau} k_n^{-\tau+1/2} r(z_n)/r(x_n) \to 0. \quad (33)$$

Next, we prove (33).

In case $F \in D(\varphi_\alpha)$ we have $r(z_n)/r(x_n) \to 1$ and (33) follows directly from assumption H_1.

Let $F \in D(\Lambda)$. Then $G \in D(\Lambda)$ and by (27) and Lemma 1,

$$l_n^{\tau} k_n^{-\tau+1/2} r(z_n)/r(x_n) \leq 2 l_n^{\tau-\epsilon} k_n^{\tau-1/2-\epsilon},$$

where ϵ is small enough, which tends to 0 by H_1.

In case $F \in D(\psi_\gamma)$, for all $\zeta < 1/\gamma$, we have

$$\frac{y_0 - z_n}{y_0 - x_n} \leq 2 \sup\left(\frac{nU_{l_n+1,n}}{l_n}, 1\right)\left(\frac{l_n}{k_n}\right)^{-\zeta+1/\gamma}.$$

Using the fact that $r(z_n) \sim \frac{y_0 - z_n}{\gamma+1}$, we get the desired result. This in turn gives

$$R_n(4) \to_p 0.$$

Next, by the fact that $nU_{k_n+1,n}/k_n \to_p 1$ whenever $k_n \to \infty$ with $k_n/n \to 0$, we have

$$D_n = (1 + o_p(1)) \frac{k_n^{1/2}}{r(1, x_n)} (\widetilde{x}_n - x_n). \quad (34)$$

In case $F \in D(\Lambda)$, using the representation (3) and the fact that $r(1, x_n) \sim S(k_n/n)$, we have

$$\frac{k_n^{1/2}}{r(1, x_n)} (\widetilde{x}_n - x_n) = n k_n^{-1/2}\left(U_{k_n+1,n} - \frac{k_n}{n}\right)(1 + o_p(1)) + k_n^{1/2}\left(\frac{S(U_{k_n+1,n})}{S(\frac{k_n}{n})} - 1\right).$$

The first term of this equation, which we denote by $N(2)$, is Gaussian and the second term goes to 0 in probability as $k_n^{1/2}(f(U_{k_n+1,n}) - f(\frac{k_n}{n}))$ tends to 0 in probability.

When $F \in D(\varphi_\alpha)$, this case is exactly the previous one, since $G \in D(\Lambda)$ and the representation (1) holds.

Let $F \in D(\Psi_\gamma)$. Use $r(1, x_n) \sim (y_0 - x_n)/\gamma + 1$ and get

$$\frac{\sqrt{k_n}}{r(1, x_n)} (\widetilde{x}_n - x_n) \sim (\gamma + 1) k_n^{1/2}\left(\frac{y_0 - \widetilde{x}_n}{y_0 - x_n} - 1\right).$$

Now by (1),

$$(1 + \gamma_n(k_n))\left(\frac{n}{k_n} U_{k_n+1,n}\right)^{-\varepsilon_n+1/\gamma} - 1 \leq \frac{y_0 - \widetilde{x}_n}{y_0 - x_n} - 1 \leq (1 + \gamma_n(k_n))\left(\frac{n}{k_n} U_{k_n+1,n}\right)^{\varepsilon_n+1/\gamma} - 1, \quad (35)$$

where $\varepsilon_n = \sup\{b(t), t \leq \max(U_{k_n+1,n}, k_n/n)\} \xrightarrow{p} 0$. Since $\frac{n}{k_n} U_{k_n+1,n} \xrightarrow{p} 1$,

$$(\gamma + 1) k_n^{1/2}\left(\frac{y_0 - \widetilde{x}_n}{y_0 - x_n} - 1\right) = (\gamma + 1) k_n^{1/2} \gamma_n(k_n)(1 + o_p(1))$$

$$+ \frac{\gamma + 1}{\gamma} n k_n^{-1/2}(U_{k_n+1,n} - k_n/n)(1 + o_p(1)).$$

Put $N(2) = \frac{\gamma+1}{\gamma} n k_n^{-1/2}(U_{k_n+1,n} - k_n/n)$ and $N(3,\tau) = N(1,\tau) + N(2)$. It is easy to see that the asymptotic form of $\mathbb{E}(N(2)^2)$ is $(\frac{\gamma+1}{\gamma})^2$. Further, the covariance between $N(1,\tau)$ and $N(2)$ is

$$-\frac{\tau(\gamma+1)}{r(1,x_n)\gamma}\left(\frac{n}{k_n}\right)^{1/2}\left(\frac{n}{k_n}\right)^{\tau-1/2}\int_{x_n}^{z_n}(1-G(t))^{\tau-1}\left[\min\left(\frac{k_n}{n}, 1-G(t)\right) - \frac{k_n}{n}(1-G(t))\right] dt$$

$$\sim -\frac{\tau(\gamma+1)}{r(1,x_n)\gamma}\left(\frac{n}{k}\right)^{\tau}\int_{x_n}^{z_n}(1-G(t))^{\tau}\, dt \sim -\frac{\tau(\gamma+1)}{r(1,x_n)\gamma} \sim -\frac{\tau(\gamma+1)^2}{\gamma(\gamma\tau+1)}.$$

Hence, $N(3,\tau)$ is a normal random variable with variance $\sigma_2^2(\tau)$. This, (31), (32), and (34) together establish the characterization given in Part 1 of Theorem 2. We have now the means to evaluate the covariance between $N_n(3,\tau)$ and $N_n(3,\rho)$) and then $\Gamma_2(\tau,\rho)$, which is

$$\Gamma_2(\tau,\rho) = \Gamma_1(\tau,\rho) - \frac{\gamma\tau(\gamma+1)^2}{\gamma^2(\gamma\tau+1)} - \frac{\tau\rho(\gamma+1)^2}{\gamma^2(\gamma\tau+1)} + \left(\frac{\gamma+1}{\gamma}\right)^2.$$

It remains to get the law of $T_n(\tau)/T_n(\rho)$ under (RC). The same arguments as used in (30) yield

$$\begin{aligned} k_n^{1/2}\left(\frac{T_n(\tau)}{T_n(\rho)} - \frac{\mu_n(\tau)}{\mu_n(\rho)}\right) &= \frac{(\gamma\rho+1)N(3,\tau)}{(\gamma+1)} - \frac{(\gamma\rho+1)^2}{(\gamma+1)(\gamma\tau+1)}N(3,\rho) + o_p(1) \\ &=: N(5,\tau,\rho) + o_p(1). \end{aligned}$$

An easy computation enables us to get

$$\begin{aligned} \mathbb{E}(N(5,\tau,\rho)^2) = \gamma^{-2}\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2 \Bigg\{ &\frac{(\tau\gamma^2-\gamma+2)(\gamma\tau+1)}{\gamma(2\tau-1)+2} + \frac{(\rho\gamma^2-\gamma+2)(\gamma\rho+1)}{\gamma(2\rho-1)+2} \\ &- 2\frac{\gamma^3\tau\rho+\gamma(\tau+\rho-1)+2}{\gamma(\rho+\tau-1)+2}\Bigg\} = \sigma_4^2(\tau,\rho). \end{aligned}$$

This achieves completely the proof of the theorem. □

Proof of Theorem 3. We give the proof of (12). The proof of (13) is similar to the proof of (11) in Theorem 2 and hence is omitted.

First let us give the following decomposition:

$$\begin{aligned} \frac{k_n^{1/2}}{r(x_n)}\left[T_n(\tau) - \frac{y_0-x_n}{\gamma\tau+1}\right] &= \frac{k_n^{1/2}}{r(x_n)}[T_n(\tau)-\mu_n(\tau)] + \frac{k_n^{1/2}}{r(x_n)}[\mu_n(\tau) - r(\tau,x_n,z_n)] \\ &+ \frac{k_n^{1/2}}{r(x_n)}\left[r(\tau,x_n,z_n) - \frac{y_0-x_n}{\gamma\tau+1}\right] =: A+B+C. \qquad (36) \end{aligned}$$

By Theorem 2, A converges in distribution to a centered normal random variable with variance $\sigma_2^2(\tau)$. Next we notice that $\mu_n(\tau) = \left[\frac{n}{k_n}(1-G(x_n))\right]^{\tau} r(\tau,x_n,z_n)$. Then

$$B = k_n^{1/2}\frac{r(\tau,x_n,z_n)}{r(x_n)}\left[\left(\frac{n}{k_n}(1-G(x_n))\right)^{\tau} - 1\right].$$

Using Lemma 1, we can easily prove that $\frac{r(\tau,x_n,z_n)}{r(x_n)}$ tends to $\frac{1}{\gamma\tau+1}$ and we conclude that $B \to 0$ by H_3. Il remains now to prove that $C \to 0$.

Using the Karamata representation of G given in (22), we can write

$$r(\tau,x,z) = (y_0-x)^{-\gamma\tau}\int_x^z (y_0-t)^{\gamma\tau}\left(\frac{1+f(t)}{1+f(x)}\right)^{\tau}\exp\left(\tau\int_x^t\frac{b(s)}{y_0-s}\,ds\right)dt. \qquad (37)$$

But

$$\left(\frac{1+f(t)}{1+f(x)}\right)^{\tau} = 1 + O(\delta(x)) \qquad \text{uniformly in} \quad t \in (x, y_0), \tag{38}$$

where

$$\delta(x) = \sup\{|f(t)|, x \le z \le y_0\} \to 0 \qquad \text{as} \quad x \to y_0.$$

From (37) and (38), we have

$$r(\tau, x, z) = (1 + O(\delta(x)))(y_0 - x)^{-\gamma\tau} \int_x^z (y_0 - t)^{\gamma\tau} \exp\left(\tau \int_x^t \frac{b(s)}{y_0 - s}\, ds\right) dt.$$

Put the variable change $y_0 - t = p$ and get

$$r(\tau, x, z) = (1 + O(\delta(x)))(y_0 - x)^{-\gamma\tau} \int_{y_0 - z}^{y_0 - x} p^{\gamma\tau} \exp\left(\tau \int_x^{y_0 - p} \frac{b(s)}{y_0 - s}\, ds\right) dp.$$

Set $\lambda < 1$ and put h the integer such that

$$(y_0 - x)\lambda^{h+1} < y_0 - z \le (y_0 - x)\lambda^h.$$

Denote

$$a_0 = y_0 - z, \qquad a_j = (y_0 - x)\lambda^{h-j+1}, \quad j = 1, \dots, h+1.$$

Then

$$\begin{aligned}
\int_{y_0 - z}^{y_0 - x} p^{\gamma\tau} \exp\left(\tau \int_x^{y_0 - p} \frac{b(s)}{y_0 - s}\, ds\right) dt &= \sum_{j=0}^{h} \int_{a_j}^{a_{j+1}} p^{\gamma\tau} \exp\left(\tau \int_x^{y_0 - p} \frac{b(s)}{y_0 - s}\, ds\right) dp \\
&\le \sum_{j=0}^{h} \int_{a_j}^{a_{j+1}} p^{\gamma\tau} \exp\left(\tau\beta(x) \int_x^{y_0 - p} \frac{ds}{y_0 - s}\, ds\right) dp \\
&= \sum_{j=0}^{h} \int_{a_j}^{a_{j+1}} p^{\gamma\tau} \exp\left(-\tau\beta(x) \log\left(\frac{p}{y_0 - x}\right)\right) dp \\
&\le \sum_{j=0}^{h} \int_{a_j}^{a_{j+1}} p^{\gamma\tau} \exp\big(-\tau\beta(x)(j+1)\log\lambda\big)\, dp \\
&= \exp\big(-\tau\beta(x)(h+1)\log\lambda\big) \frac{1}{\gamma\tau + 1} \sum_{j=0}^{h} (a_{j+1}^{\gamma\tau+1} - a_j^{\gamma\tau+1}) \\
&= \exp\big(-\tau\beta(x)(h+1)\log\lambda\big) \frac{1}{\gamma\tau + 1} \big((y_0 - x)^{\gamma\tau+1} - (y_0 - z)^{\gamma\tau+1}\big),
\end{aligned}$$

where $\beta(x) = \sup\{|b(t)|, x \le z \le y_0\} \to 0$ as $x \to y_0$.

Finally,

$$r(\tau, x, z) \le (1 + O(\delta(x))) \frac{y_0 - x}{\gamma\tau + 1}\left(1 - \left(\frac{y_0 - z}{y_0 - x}\right)^{\gamma\tau+1}\right) \exp\big(-\tau\beta(x)(h+1)\log\lambda\big).$$

This holds for any $\lambda < 1$. We get while $\lambda \uparrow 1$

$$r(\tau, x, z) \le (1 + O(\delta(x))) \frac{y_0 - x}{\gamma\tau + 1}\left(1 - \left(\frac{y_0 - z}{y_0 - x}\right)^{\gamma\tau+1}\right).$$

By similar arguments, we obtain the same kind of lower bound, so that

$$|C| \leq \frac{y_0 - x_n}{(\gamma\tau + 1)r(x_n)} k_n^{1/2} \left(\frac{y_0 - z_n}{y_0 - x_n}\right)^{\gamma\tau+1}$$
$$+ O(k_n^{1/2}\delta(x_n)) \frac{y_0 - x_n}{(\gamma\tau + 1)r(x_n)} \left[1 - \left(\frac{y_0 - z_n}{y_0 - x_n}\right)^{\gamma\tau+1}\right] = C_1 + C_2.$$

Since $F \in D(\psi_\gamma)$, we have for all ζ small enough such that $\zeta < 1/\gamma$,

$$\frac{y_0 - z_n}{y_0 - x_n} \leq 2 \sup \left(\frac{nU_{l_n+1,n}}{l_n}, 1\right) \left(\frac{l_n}{k_n}\right)^{-\zeta+1/\gamma},$$
$$k_n^{1/2} \left(\frac{y_0 - z_n}{y_0 - x_n}\right)^{\gamma\tau+1} \leq C' k_n^{1/2} \left(\frac{l_n}{k_n}\right)^{(-\zeta+1/\gamma)(\gamma\tau+1)},$$

where C' is a nonnegative constant. Hence by H_1 and using the fact that $r(x_n) \sim \frac{y_0 - x_n}{\gamma+1}$, C_1 tends to 0. The term C_2 tends to 0 by H_2. □

5. CONCLUSION

The remaining case $\tau \leq 1/2$ is to be studied later with more restrictive conditions. The continuous asymptotic convergence of these processes in $C([T, T'])$ spaces, $1/2 < T < T'$, will be studied in future research works.

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